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BASIC CALCULUS

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PREFACE

This worktext on Basic Calculus is designed particularly for senior high school students under the K – 12 Basic Education Curriculum. The contents are parallel to the learning competency as indicated in the curriculum guide for Basic Calculus. It also aims to prepare them for higher education which is one of the thrusts of the K – 12 Basic Education Curriculum.

In every chapter , there is the introduction of mathematical concepts and followed by several illustrative examples. This provides the students better understanding and appreciation of the lessons. Sufficient exercises for home studies are provided in each chapter to further strengthen and test the students' understanding of the topics. Hence, students are given the opportunity to assess their understanding of the concepts.

There are three chapters presented in this worktext:

- Chapter 1 – Limits and Continuity of Functions
- Chapter 2 – Derivatives of Algebraic Functions
- Chapter 3 – Antidifferentiation

Criticisms and suggestions made by mathematics faculty members and students have been incorporated in the worktext. However, flaws and errors are still to be seen. The authors will be highly grateful and appreciative if they will be given feedback regrading this concern.

THE AUTHORS

CHAPTER 1
LIMITS AND CONTINUITY OF FUNCTIONS

CHAPTER 2
DERIVATIVES OF ALGEBRAIC FUNCTIONS

CHAPTER 3
ANTIDIFFERENTIATION

C

CHAPTER 1

LIMITS AND CONTINUITY OF FUNCTIONS

Learning Competencies

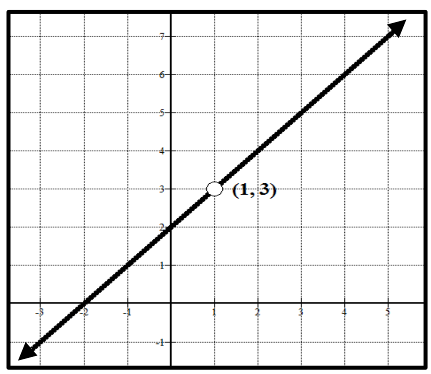
At the end of the chapter, the students learn to:

- Illustrate the limit of a function using a table of values and the graph of the function.
- Distinguish between $\lim_{x \to a} f(x)$ and $f(a)$.
- Illustrate the limit laws.
- Apply the limit laws in evaluating the limit of algebraic functions (polynomial, rational, radical).
- Compute the limits of exponential, logarithmic and trigonometric functions using table of values and graphs of the functions.
- Evaluate limits involving $\frac{\sin x}{x}$, $\frac{1-\cos x}{x}$ and $\frac{e^x-1}{x}$ using table of values.
- Illustrate continuity of a function at a number.
- Determine whether a function is continuous at a number or not.
- Illustrate continuity of a function on an interval.
- Determine whether a function is continuous on an interval or not.
- Illustrate different types of discontinuity (removable, jump, infinite).
- Illustrate the Intermediate Value and Extreme Value Theorems.
- Solve problems involving continuity of a function.

1.1. Limit of a Function

In Calculus, the idea of limit is very important. The concept of limit is at the foundation of almost all mathematical analysis and an understanding of it is absolutely essential. Deep understanding of limit is very rewarding since it facilitates a good grasp of all the basic processes of Calculus.

Let us consider a particular function, say $f(x)=\dfrac{x^2+x-2}{x-1}$. This function is defined for all values of x except at $x=1$. This is because both the numerator and denominator of function $f(x)$ take zero value at $x=1$, that is, $f(1)=\dfrac{0}{0}$, a meaningless expression or is an indeterminate form, hence, making one not an element of the domain of $f(x)$. However, factoring $f(x)=\dfrac{x^2+x-2}{x-1}$ reduces it to $f(x)=\dfrac{(x+2)(x-1)}{x-1}=x+2$, provided $x\neq 1$. Note on the straight line graph of $f(x)=y=x+2$ shown below, point $(1,3)$ does not lie on the graph, hence, an open circle or a hole is seen on its graph.



Let us study how the function f behaves when we assume values of x getting closer and closer to 1. This means x nears 1 but never equal to 1. There are two ways by which value of x may approach 1. One is by assuming values of x less than 1 (or at the left of 1) but approaching 1. The other way is by assuming values of x more than 1 (or at the right of 1) but also approaching 1.

To get a better view of what happens as x takes values approaching 1 but less than 1 $(x<1)$, let us consider the table of values on the succeeding discussions. From Table 1 on the next page, observe that the value of function $f(x)=\dfrac{x^2+x-2}{x-1}$ approaches 3 as x takes values less than 1 but approaching 1. Note that as x gets closer and closer to 1 through values less than 1, the value of $f(x)$ gets closer and closer to 3; and the closer x is to 1, the closer $f(x)$ is to 3. We can make the value of

$f(x)$ or the y value as close to 3 as we please by taking x close enough to 1. Another way of saying this is that we can make the absolute value of the difference between $f(x)$ and 3 as small as we please by making the absolute value of the difference between x and 1 small enough. That is, $|f(x)-3|$ can be made as small as we please by making $|x-1|$ small enough. But bear in mind that $f(x)$ never takes on the value 3.That is, provided $x \neq 1$, as $x \to 1^-$, $f(x) \to 3$. This is read "as x approaches 1 through values less than 1, $f(x)$ approaches 3". In symbol form, $\lim_{x \to 1^-} f(x) = 3$ read "limit of $f(x)$ as x approaches 1 through values less than 1 is equal to 3". In particular, this limit is called the *Left-hand Limit* of $f(x)$.

Table 1

x	$f(x)=\dfrac{x^2+x-2}{x-1}$
0.90	2.9
0.99	2.99
0.999	2.999
0.9999	2.9999
0.99999	2.99999
0.999999	2.999999
0.9999999	2.9999999

1.2. Definition of Left-hand Limit of a Function

The notation for the Left-Hand Limit of $f(x)$ is $\lim_{x \to a^-} f(x) = L$.

Here, $f(x)$ is made close to L for all x sufficiently close to a and $x < a$ without actually letting x be a.

Let us now look at Table 2 where the values taken by x approach 1 through values greater than 1 $(x > 1)$.

Table 2

x	$f(x)=\dfrac{x^2+x-2}{x-1}$
1.1	3.1
1.01	3.01
1.001	3.001
1.0001	3.0001
1.00001	3.00001
1.000001	3.000001
1.0000001	3.0000001

Observe that the value of the function f gets closer and closer to 3 but not equal to 3. That is, when $x \to 1^+$, $f(x) \to 3$, provided $x \neq 1$. Symbolically, $\lim_{x \to 1^+} f(x) =$ 3. This is read "the limit of $f(x)$ as x approaches 1 through values greater than 1 is equal to 3". This limit of $f(x)$ as $x \to 1^+$ is specifically called the *Right-hand Limit* of $f(x)$. The Right-hand and Left-hand Limits are referred to as One-sided Limit of the function.

1.3. Definition of Right-hand Limit of a Function

> The notation for the Right-Hand Limit of $f(x)$ is $\lim_{x \to a^+} f(x) = L$
>
> Here, $f(x)$ is made close to a for all x sufficiently close to a and $x > a$ without actually letting x be a.

Note that the change in notation is very minor and in fact might be missed if one is not paying attention. The only difference is the bit that is under the "lim" part of the limit. For the right-hand limit, we have $x \to a^+$ (note the "$+$") which means that we need to look only at $x > a$. Likewise for the left-hand limit, we have $x \to a^-$ (note the "$-$") which means that we will only be looking at $x < a$.

1.4. Definition of Two - Sided Limit of a Function

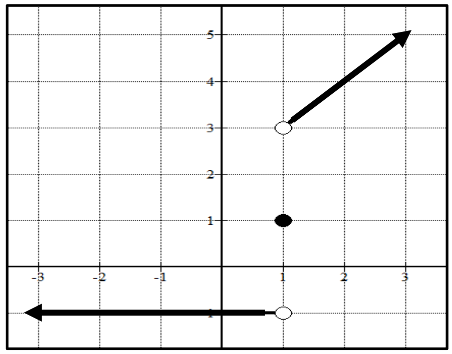
We had defined already the one-sided limit of a function. However, limit of a function is often called **two-sided limit** and this exists if both one-sided limits exist and are equal. That is,

> $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$, then, $\lim_{x \to a} f(x) = L$

The notation for the two-sided limit of $f(x)$ as $x \to a$ is $\lim_{x \to a} f(x)$. Observe that on the above notation, a does not bear anymore the superscript $+$ or $-$. Similarly, note that on our illustrative example, the given function $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ has both $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 3$. Therefore, $\lim_{x \to 1} f(x) = 3$.

But this may not always be the case in other functions where the left-hand limit and right-hand limit are different in values. Hence, we say that the limit of the function does not exist which we may simply denote by DNE.

Consider the given graph of a certain piecewise-defined function $h(x)$ below.



Observe from the graph the following properties of function $h(x)$.

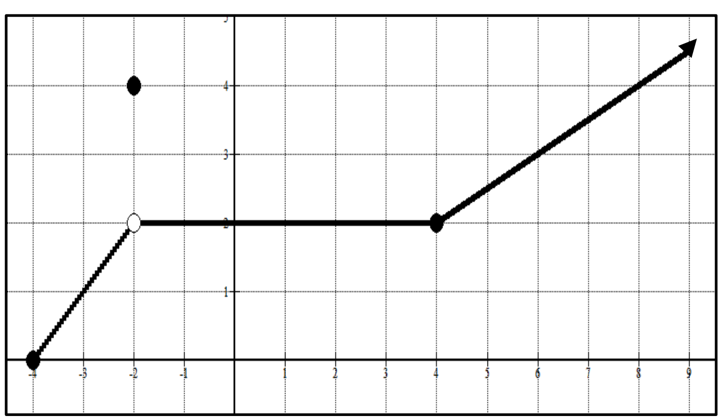
- $\lim_{x \to 1^+} h(x) = 3$
- $\lim_{x \to 1^-} h(x) = -1$
- $h(1) = 1$

From the above-listed properties, we conclude that limit of $h(x)$ as $x \to 1$ does not exist since $\lim_{x \to 1^+} h(x) \neq \lim_{x \to 1^-} h(x)$. Hence, the $\lim_{x \to 1} h(x) = DNE$.

1.5. Difference between $\lim_{x \to a} f(x)$ and Function Value $f(a)$.

From the previous discussion, it is apparent that the function $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ can be made as close to 3 as we please by taking x sufficiently close to 1. However, this property of the function f does not depend on f being defined when $x = 1$. This fact gives the distinction between limit of $f(x)$ as x approaches 1 and the function value at $x = 1$.

Let us again consider the graph of function $g(x)$ shown below.



From the preceding graph, we can enumerate some of its properties.

- $\lim_{x \to -2^-} g(x) = \lim_{x \to -2^+} g(x) = \lim_{x \to -2} g(x) = 2$
- $g(-2) = 4$
- $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x) = \lim_{x \to 4} g(x) = 2$
- $g(4) = 2$

Note that $\lim_{x \to -2} g(x) \neq g(-2)$ while $\lim_{x \to 4} g(x) = g(4) = 2$. This example shows that even if the function g is defined at $x = a$, it is possible for the limit of the function g to exist even without having the same value for $g(a)$.

1.6. Theorems on Limit of a Function

Let $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$.

1) $\lim_{x \to a} c = c$, where $c \in R$
2) $\lim_{x \to a} x = a$
3) $\lim_{x \to a} [f(x) \pm g(x)] = A \pm B$
4) $\lim_{x \to a} [f(x) \bullet g(x)] = A \bullet B$
5) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{A}{B}$, *provided* $B \neq 0$
6) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = [A]^n$
7) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$

Example 1. Evaluate the following limits by using the laws on limit of a function.

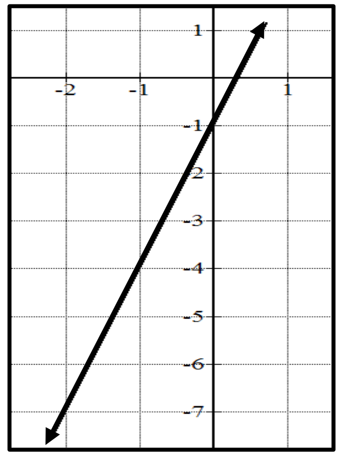
a. $\lim_{x \to -2} (3x - 1)$

Use Theorem 1. $\lim_{x \to -2} (3x - 1) = \lim_{x \to -2} 3x - \lim_{x \to -2} 1$

Use Theorems 1 and 4. $\lim_{x \to -2} (3x - 1) = \lim_{x \to -2} 3 \bullet \lim_{x \to -2} x - \lim_{x \to -2} 1$

Use Theorems 1 and 2. $\lim_{x \to -2} (3x - 1) = 3(-2) - 1 = -6 - 1 = -7$

The graph of function $y = 3x - 1$ shows that as $x \to -2$, the y value approaches value of -7. Therefore, $\lim_{x \to -2} (3x - 1) = -7$.



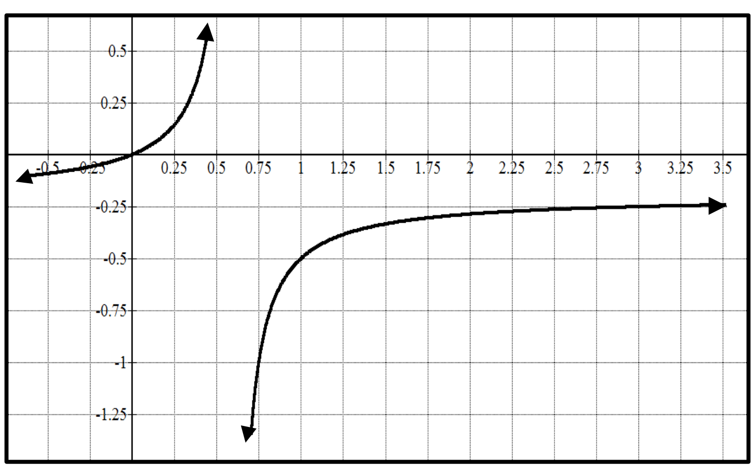
b. $\lim_{x \to 3} \frac{x}{3 - 5x}$

Use Theorem 5. $\lim_{x \to 3} \frac{x}{3 - 5x} = \frac{\lim_{x \to 3} x}{\lim_{x \to 3} (3 - 5x)}$

Use Theorems 2 and 3. $\lim_{x \to 3} \frac{x}{3 - 5x} = \frac{3}{\lim_{x \to 3} 3 - \lim_{x \to 3} 5x}$

Use Theorems 1 and 4. $\lim_{x \to 3} \frac{x}{3 - 5x} = \frac{3}{3 - \lim_{x \to 3} 5 \cdot \lim_{x \to 3} x} = \frac{3}{3 - 5(3)} = \frac{3}{3 - 15} = \frac{3}{-12} = -\frac{1}{4}$

The graph of function $y = \frac{x}{3-5x}$ below shows that as $x \to 3$, the y value approaches $-\frac{1}{4}$. Hence, $\lim_{x \to 3} \frac{x}{3 - 5x} = -\frac{1}{4}$.



c. $\lim_{x\to1}\sqrt{\dfrac{4-x^2}{1+x^3}}$

Use Theorem 7. $$\lim_{x\to1}\sqrt{\frac{4-x^2}{1+x^3}}=\sqrt{\lim_{x\to1}\frac{4-x^2}{1+x^3}}$$

Use Theorem 5 and 3. $$\lim_{x\to1}\sqrt{\frac{4-x^2}{1+x^3}}=\sqrt{\frac{\lim_{x\to1}(4-x^2)}{\lim_{x\to1}(1+x^3)}}=\sqrt{\frac{\left[\lim_{x\to1}4-\lim_{x\to1}x^2\right]}{\left[\lim_{x\to1}1+\lim_{x\to1}x^3\right]}}$$

Use Theorems 1 and 6. $$\lim_{x\to1}\sqrt{\frac{4-x^2}{1+x^3}}=\sqrt{\frac{4-\left(\lim_{x\to1}x\right)^2}{1+\left(\lim_{x\to1}x\right)^3}}=\sqrt{\frac{4-(1)^2}{1+(1)^3}}=\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}$$

Take note that limits of the given functions presented can be achieved by using direct substitution method where the value approached by the variable is substituted on the given function.

Example 2. Evaluate the following limits by using the direct substitution method.

a). $\lim_{x\to2}\dfrac{3x^2-x-10}{x^2-4}$

Solution:

To evaluate the limit of the given function, using the direct substitution method, replace 2 for x.

$$\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}=\frac{0}{0}\quad\text{(An Indeterminate Form)}$$

Simplify the given function to eliminate the factor common to both the numerator and denominator and that has zero value at $x=2$.

$$\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

Divide out the factor $(x-2)$ which is causing the indeterminate form, then, evaluate the limit.

$$\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to2}\frac{3x+5}{x+2}=\frac{3(2)+5}{2+2}=\frac{11}{4}$$

b). $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Solution:

By using the direct substitution method, replace the variable x by 3 to evaluate the limit of the given function.

$$\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\frac{0}{0}\quad \text{(An Indeterminate Form)}$$

The factor $x=3$ which is zero when $x=3$ needs to be eliminated from the numerator and denominator of the given function.

$$=\lim_{x\to 3}\frac{(x^2-9)(x^2+9)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$

$$=\frac{(3+3)[(3)^2+9]}{2(3)+1}=\frac{6(9+9)}{7}=\frac{6(18)}{7}=\frac{108}{7}$$

c. $\lim_{x\to 4}\frac{3-\sqrt{x+5}}{x-4}$

Solution:

By using the direct substitution method, replace the variable x by 4 to evaluate the limit of the given function.

$$\lim_{x\to 4}\frac{3-\sqrt{x+5}}{x-4}=\frac{0}{0}\quad \text{(An Indeterminate Form)}$$

Factor $(x-4)$ must be removed from the numerator and denominator. To do it, multiply the members of the fraction by the conjugate of the numerator to eliminate the radical of index two. Then, recall and apply the product of a sum and difference of two terms: $(a+b)(a-b)=a^2-b^2$.

$$=\lim_{x\to 4}\frac{3-\sqrt{x+5}}{x-4}\cdot\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}=\lim_{x\to 4}\frac{9-(x+5)}{(x-4)\left(3+\sqrt{x+5}\right)}=\lim_{x\to 4}\frac{4-x}{(x-4)\left(3+\sqrt{x+5}\right)}$$

$$=\lim_{x\to 4}\frac{-(x-4)}{(x-4)\left(3+\sqrt{x+5}\right)}=\lim_{x\to 4}\frac{-1}{3+\sqrt{x+5}}=\frac{-1}{3+3}=-\frac{1}{6}$$

d). $\lim_{x\to 0} \frac{x^3 - 7x}{x^3}$

Solution:

Direct substitution of zero for x results to $\lim_{x\to 0} \frac{x^3 - 7x}{x^3} = \frac{0}{0}$.

$$= \lim_{x\to 0} \frac{x(x^2 - 7)}{x^3} = \lim_{x\to 0} \frac{x^2 - 7}{x^2} = \frac{-7}{0} = -\frac{7}{0} = -\infty$$

Note that the numerator approaches -7 and the denominator is a positive quantity approaching 0. The quantity $-\infty$ is NOT a real number and is NOT an indeterminate form. Hence, the limit of the given function does not exist or DNE.

e). $\lim_{x\to 0} \frac{\cos 2x - 1}{cosx - 1}$

Solution:

Replacing $\cos 0^o$ by 1 gives again a limit of the form $\frac{0}{0}$. That is,

$$\lim_{x\to 0} \frac{cos2x - 1}{cosx - 1} = \frac{0}{0} \qquad \text{(An Indeterminate Form)}$$

Recall the trigonometric identity: $\cos 2x = 2\cos^2 x - 1$

Substitution into the given expression results to:

$$= \lim_{x\to 0} \frac{(2cos^2x - 1) - 1}{cosx - 1} = \lim_{x\to 0} \frac{2cos^2x - 2}{cosx - 1}$$

Factor-out 2 from the binomial numerator and the second factor $(\cos^2 x - 1)$ as difference of squares.

$$= \lim_{x\to 0} \frac{2(cos^2x - 1)}{cosx - 1} = \lim_{x\to 0} \frac{2(cosx + 1)(cosx - 1)}{cosx - 1}$$

Cancel-out the common factor$(\cos x - 1)$ that gives zero value on the numerator and denominator, then substitute $x = 0$.

$$= \lim_{x\to 0} 2(cosx + 1) = 2(1 + 1) = 4$$

1.7. Limit of a Function Involving Infinity

If we consider the function $f(x)=\frac{1}{x}$, it is an observation that as $x\to 0$ through values at the right of 0, the corresponding values of the function get bigger and bigger. We say that $f(x)$ increases without limit or $f(x)\to\infty$ as $x\to 0^{+}$.

Hence, in symbol form:

$$\lim_{x\to 0^{+}} f(x)=\infty.$$

Likewise, as $x\to 0$ through negative values, the value of the function decreases without limit.

Thus, in symbol form:

$$\lim_{x\to 0^{-}} f(x)=-\infty.$$

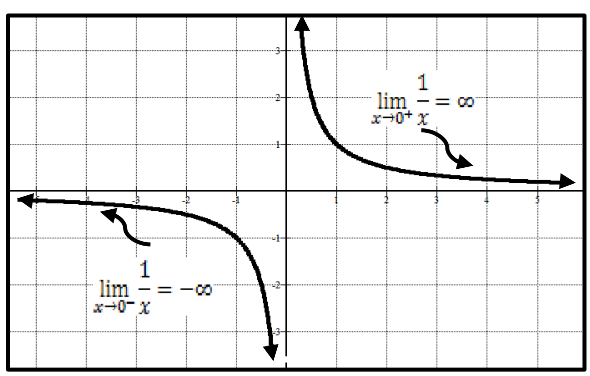
The introduction of the symbol ∞ does not in any way justify its use as a number. It is best to mention that the word "infinite" signifies only a state of being non-finite. Division by zero is a meaningless operation and it is not intended that the symbol ∞ represent $\frac{1}{0}$. Once again, it is to be stressed out that ∞ is not a symbol for a real number.

When the limit of the function as x approaches a value, say a, is infinity, we say that the limit of the function does not exist.

The symbol ∞ simply indicates the behavior of the function as x gets closer and closer to value a.

In the same manner, getting $-\infty$ for the limit of the function simply indicates that the behavior of the function whose function values decrease without bound. Getting $-\infty$ once again tells us that the limit of the function does not exist.

Example 3. Let us examine the behavior of the function $f(x)=\frac{1}{x}$ graphically shown below.



Using the above properties of $f(x)=\frac{1}{x}$,

a) $\lim_{x\to 0^{-}} 4^{\frac{1}{x}}=4^{-\infty}=\frac{1}{4^{\infty}}=\frac{1}{\infty}=0$

b) $\lim_{x\to 0^+} 4^{\frac{1}{x}} = 4^{\infty} = \infty$

Observe that as $x \to 0^-$, the function $4^{\frac{1}{x}} \to 0$ while as $x \to 0^+$, $4^{\frac{1}{x}} \to \infty$.

1.8. Theorems on Limit of a Function Involving Infinity

1) $\lim_{x\to\infty} cx = \infty \ (c > 0)$
2) $\lim_{x\to\infty} cx = -\infty \ (c < 0)$
3) $\lim_{x\to\pm\infty} \frac{c}{x} = 0$
4) $\lim_{x\to 0^+} \frac{c}{x} = \infty \ (c > 0)$
5) $\lim_{x\to 0^-} \frac{c}{x} = -\infty \ (c > 0)$

Example 4. Evaluate the limits of the following functions.

a) $\lim_{x\to\infty} \frac{4x-5}{6x+7}$

Solution:

Substitution of ∞ for x results to indeterminate $\frac{\infty}{\infty}$. In case like this, we use a standard technique in working with infinite limits by dividing each term on the numerator and denominator by the highest power of the variable x . Then, use Theorem 3 on limit of function involving infinity. Thus,

$$\lim_{x\to+\infty} \frac{4x-5}{6x+7} = \lim_{x\to+\infty} \frac{4-\frac{5}{x}}{6+\frac{7}{x}}, \left(\text{provided } x \neq 0 \text{ and } x \neq -\frac{7}{6}\right)$$

$$= \frac{4-0}{6+0} = \frac{4}{6} = \frac{2}{3}$$

b) $\lim_{x\to-\infty} \frac{4x+3}{3x^2+1}$

Solution:

The limit takes the indeterminate form $-\frac{\infty}{\infty}$. Use the technique described on the previous illustrative example by dividing both numerator and denominator by x^2, the highest power of x, and then, using Theorem 3.

$$\lim_{x\to-\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to-\infty} \frac{\frac{4}{x}+\frac{3}{x^2}}{3+\frac{1}{x^2}} = \frac{0+0}{3+0} = \frac{0}{3} = 0$$

c) $\lim_{x \to +\infty} \frac{6x^3 + x^2 + 2x - 1}{x^2 + x + 2}$

Solution:

Divide each term on the numerator and denominator by x^3, the highest power of x and then use Theorem 3 since the evaluated limit of the given function equals $\frac{\infty}{\infty}$. Hence,

$$\lim_{x \to +\infty} \frac{6x^3 + x^2 + 2x - 1}{x^2 + x + 2} = \lim_{x \to \infty} \frac{6 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x^3}}{\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3}} = \frac{6 + 0 + 0 + 0}{0 + 0 + 0} = \frac{6}{0} = \infty$$

On Example 4, observe that in evaluating limit of function of the form $\frac{f(x)}{g(x)}$ as x approaches $\pm\infty$, if:

i. The degree of the numerator equals the degree of the denominator, the limit of $\frac{f(x)}{g(x)}$ as x approaches $+\infty$ or $-\infty$ equals the ratio of the numerical coefficient of the highest power of x on the numerator to the numerical coefficient of the highest power of x on the denominator.
ii. The degree of the numerator is less than the degree of the denominator, the limit of $\frac{f(x)}{g(x)}$ as x approaches $+\infty$ or $-\infty$ equals zero.
iii. The degree of the numerator is greater than the degree of the denominator, the limit of $\frac{f(x)}{g(x)}$ as x approaches $+\infty$ or $-\infty$ equals either ∞ or $-\infty$ as the case may be.

Example 5. Examine how the limits of the following functions are evaluated.

a) $\lim_{x \to 0^+} \sqrt{x} = 0$

Note that in evaluating the limit of the given function as $x \to 0^+$, the values taken by x are all greater than zero but approaching zero.

b) $\lim_{x \to 0^-} \sqrt{x} = \text{DNE (does not exist)}$

Observe that the limit of the given function as $x \to 0^-$ does not exist since the values taken by x are all less than zero but approaching zero. Hence, the corresponding values of the given function are imaginary or not real numbers.

c) $\lim_{x \to 4^+} \frac{5}{x-4} = \infty$

As x takes values greater than 4 but approaching 4, the denominator $(x-4)$ is always greater than zero but approaching zero.

d) $\lim_{x \to 4^-} \frac{5}{x-4} = -\infty$

When x assumes values less than 4 but approaching 4, the denominator $(x-4)$ takes values less than zero but approaching zero.

1.9. Limits of Exponential, Logarithmic and Trigonometric Functions

Let us investigate how the value of the exponential, logarithmic and trigonometric functions behave as x approaches zero using table of values and the graph of the function.

A. Exponential Functions

a). $y = f(x) = e^x$

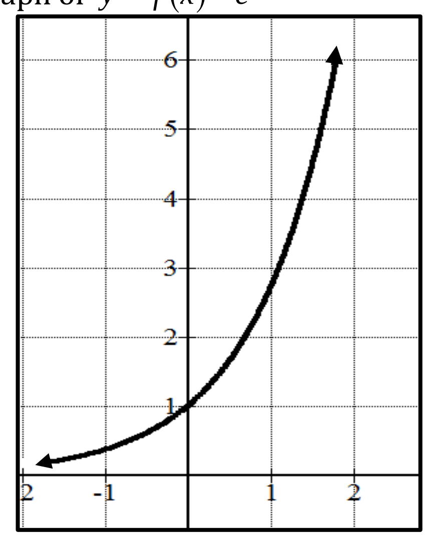
As reflected on the table of values and the graph of function $y = f(x) = e^x$ given on the next page, as $x \to 0$, $f(x) \to 1$ or the value of $y \to 1$. Hence,

$\lim_{x \to 0} e^x = 1$. Likewise, observe that the function value equals 1 when $x = 0$. That is, $f(0) = 1$. Furthermore, note that $\lim_{x \to 0} e^x = f(0) = 1$.

Table of Values

x	$y = f(x) = e^x$
0.1	1.10517084
0.01	1.01005016
0.001	1.00100050
0.001	1.00100050
0.0001	1.00010000
0.00001	1.00001000
0	1
−0.00001	0.99999000
−0.0001	0.99990001
−0.001	0.99900050
−0.01	0.99004984
−0.1	0.90483748

Graph of $y = f(x) = e^x$



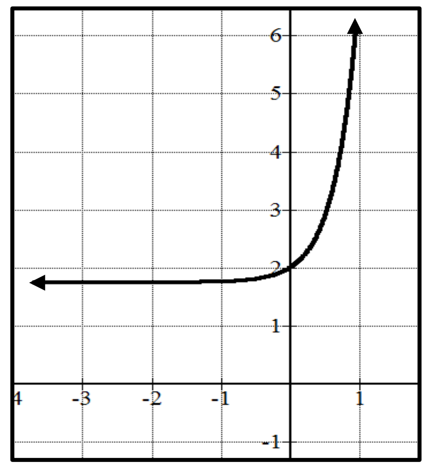
b). $y = g(x) = \dfrac{7+e^{3x}}{4}$

From the table of values and the graph of $y = g(x) = \dfrac{7+e^{3x}}{4}$ given below, observe that as $x \to 0$, $g(x) \to 2$ or the value of $y \to 2$. Therefore, $\lim_{x\to 0} g(x) = 2$. In addition, the function value at $x = 0$ equals 2. That is, $g(0) = 2$. Furthermore, take note that $\lim_{x\to 0} g(x) = g(0) = 2$.

Table of Values

x	$y = g(x) = \dfrac{7+e^{3x}}{4}$
0.01	2.0076136283
0.001	2.0007511256
0.001	2.0007511256
0.0001	2.0000750112
0.00001	2.0000075001
0	2
−0.00001	1.9999925001
−0.0001	1.9999250113
−0.001	1.9992511244
−0.01	1.9926113883

Graph of $y = g(x) = \dfrac{7+e^{3x}}{4}$



B. Logarithmic Functions

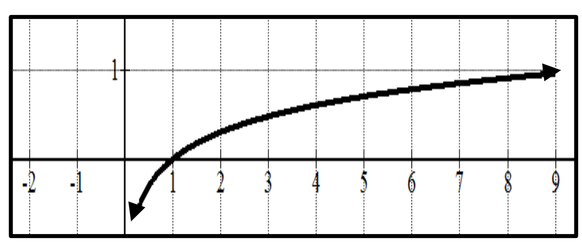
a) $y = h(x) = \log x$

The given table of values and the graph of $y = h(x) = \log x$ show that $\lim_{x\to 1} h(x) = 0$ while the function value $h(1) = 0$. Moreover, take note that $\lim_{x\to 1} h(x) = h(1) = 0$.

Table of Values

x	$y = h(x) = \log x$
1.1	0.0413927
1.01	0.0043214
1.001	0.0004341
1.0001	0.0000434
1	0
0.99999	-0.00000434290
0.9999	-0.00004343162
0.999	-0.00043451177
0.99	-0.00436480540

Graph of $y = h(x) = \log x$



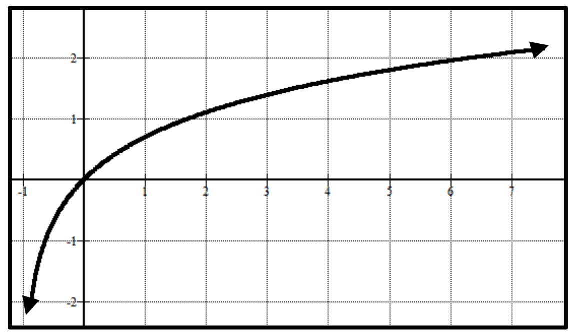
b) $y = k(x) = \ln(1+x)$

As observed from the given table of values and the graph of $y = k(x) = \ln(1+x)$ below, as $x \to 0$, $y \to 0$ or $k(x) \to 0$ which implies that $\lim_{x\to 0} k(x) = 0$ .Similarly, $k(0) = 0$. Hence, for $y = k(x) = \ln(1+x)$, $\lim_{x\to 0} k(x) = k(0) = 0$.

Table of Values

x	$y = k(x) = \ln(1+x)$
0.1	0.09531017
0.01	0.00995033
0.001	0.00099950
0	0
−0.001	−0.0010000
−0.01	−0.0100000
−0.1	−0.1053605

Graph of $y = k(x) = \ln(1+x)$



C. Trigonometric Functions

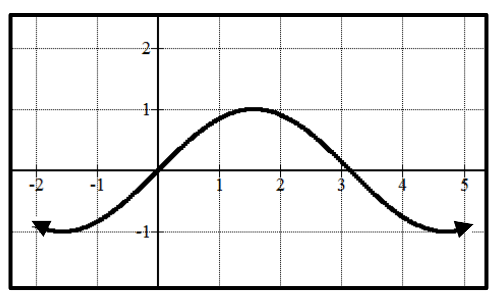
a) $y = d(x) = \sin x$

Note that from the given table of values and the graph of $y = d(x) = \sin x$ $\lim_{x\to 0} d(x) = 0$ and $d(0) = 0$. Hence, $\lim_{x\to 0} d(x) = d(0) = 0$.

Table of Values

x	$y = d(x) = \sin x$
0.1	0.099833
0.01	0.009999
0.001	0.000999
0	0
−0.001	−0.000999
−0.01	−0.009999
−0.1	−0.099833

Graph of $y = d(x) = \sin x$



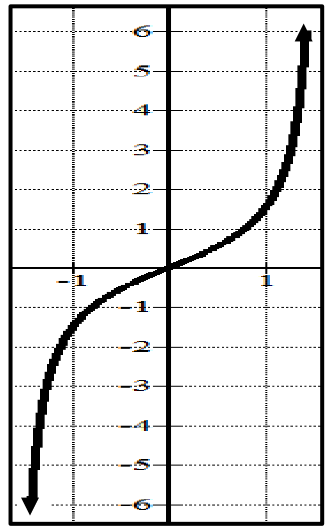
b) $y = t(x) = \tan x$

Observe that for the function$y = t(x) = \tan x$, $\lim_{x \to 0} t(x) = t(0) = 0$.

Table of Values

x	$y = t(x) = \tan x$
0.1	0.1003346
0.01	0.0100003
0.001	0.0010000
0.0001	0.0001000
0.00001	0.0000100
0	0
−0.00001	−0.0000100
−0.0001	−0.0001000
−0.001	−0.0010000
−0.01	−0.0100003
−0.1	−0.1003346

Graph of $y = t(x) = \tan x$



1.10. Limits of Functions $\frac{\sin x}{x}$, $\frac{1-\cos x}{x}$ and $\frac{e^x - 1}{x}$

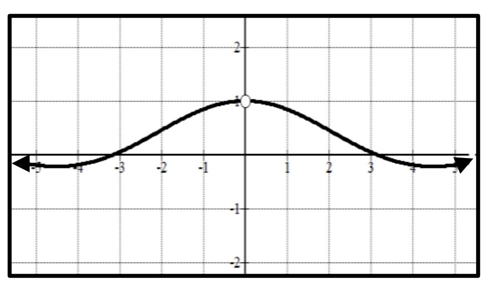
a) $y = f(x) = \frac{\sin x}{x}$

As reflected on the given table of values and the graph of function $y = f(x) = \frac{\sin x}{x}$, the $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and the function value $f(0) = \frac{0}{0}$ which is an indeterminate. Therefore, $\lim_{x \to 0} f(x) \neq f(0)$ and we see an open circle around point (0,1) indicating that the point does not lie on the graph of $y = f(x) = \frac{\sin x}{x}$

Table of Values

x	$y = f(x) = \frac{\sin x}{x}$
0.1	0.9983341
0.01	0.9999833
0.001	0.9999998
0.0001	0.9999999
0	0/0
−0.0001	0.9999999
−0.001	0.9999998
−0.01	0.9999833
−0.1	0.9983341

Graph of $y = f(x) = \frac{\sin x}{x}$



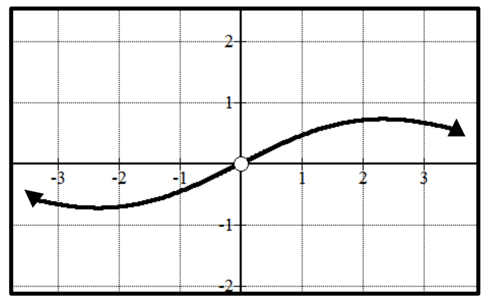
b) $y = g(x) = \frac{1-\cos x}{x}$

The given table of values and the graph of $y = g(x) = \frac{1-\cos x}{x}$ on the next page show that $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$ while $g(0) = \frac{0}{0}$ which is an indeterminate. Hence, $\lim_{x \to 0} \frac{1-\cos x}{x} \neq g(0)$ and this implies that the point $(0, 0)$ is not a point on the graph of $y = \frac{1-\cos x}{x}$ as indicated by the open circle on the graph.

Table of Values

x	$y = g(x) = \frac{1 - \cos x}{x}$
0.1	0.04995834
0.01	0.00499995
0.001	0.00049999
0.0001	0.00005000
0	0/0
−0.0001	−0.00005000
−0.001	−0.00049999
−0.01	−0.00499995
−0.1	−0.04995834

Graph of $y = g(x) = \frac{1-\cos x}{x}$



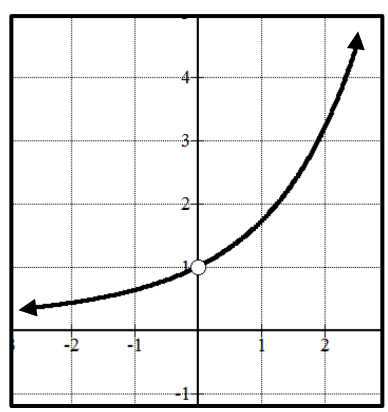
c) $y = h(x) = \frac{e^x - 1}{x}$

Just like functions $f(x) = \frac{\sin x}{x}$ and $g(x) = \frac{1-\cos x}{x}$, the function value $h(0) = \frac{0}{0}$ while $\lim_{x \to 0} h(x) = 1$. Observe that $\lim_{x \to 0} h(x) \neq h(0)$, hence an open circle is drawn at point(0,1).

Table of Values

x	$y = h(x) = \frac{e^x - 1}{x}$
0.1	1.0517091
0.01	1.0050167
0.001	1.0005001
0.0001	1.0000500
0	0/0
−0.0001	0.9999500
−0.001	0.9995001
−0.01	0.9950166
−0.1	0.9516259

Graph of $y = h(x) = \frac{e^x - 1}{x}$



1.11. Continuity of a Function at a Number

The function $f(x)$ is continuous at point $x = a$ if the following three conditions are satisfied:

1) $f(a)$ is defined,
2) $\lim_{x \to a} f(x)$ exists (i. e., is finite), and,
3) $\lim_{x \to a} f(x) = f(a)$

The graph of a continuous function has no gaps or jumps in it, meaning, we can draw the graph of a continuous function without lifting our pen from the paper.

Example 6. Determine whether the given function is continuous or not at the indicated value of x.

a) $f(x) = \frac{x^2-9}{x-3}$, at $x = 3$

b) $h(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$

c) $k(x) = \begin{cases} \frac{1}{x}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

Solution:

a) At $x = 3$, $f(x) = \frac{x^2-9}{x-3} = \frac{0}{0}$ which is an indeterminate. Therefore, x = 3 is a point of discontinuity. So, we need to eliminate the factor that causes zero value on both the numerator and denominator. We do this using the factoring method as shown below.

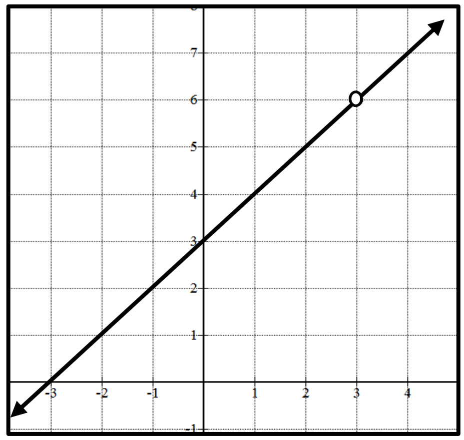
$$f(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3} = x + 3$$
$$f(3) = 3 + 3 = 6 \text{ (defined)}$$

There is an open circle on the graph of $f(x) = \frac{x^2-9}{x-3}$ at point (3, 6).

To evaluate the limit of $f(x)$ as $x \to 3$, use now the direct substitution method. That is

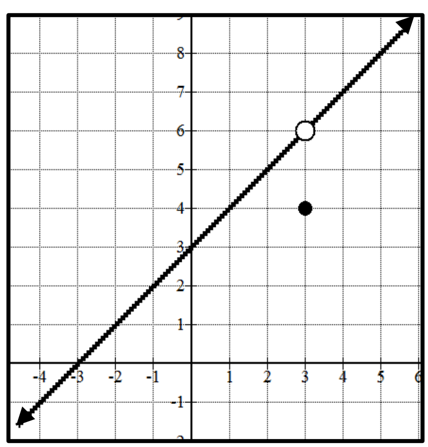
$$\lim_{x \to 3} (x + 3) = 3 + 3 = 6 \text{ (finite)}$$

And, since $\lim_{x \to 3} f(x) = f(3) = 6$, we say that the function $f(x) = \frac{x^2-9}{x-3}$ is discontinuous at $x = 3$ as seen on the graph of $f(x)$.



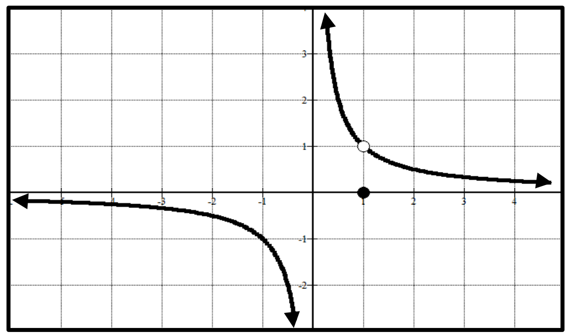
b). $h(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$

The given piecewise-defined function has $h(3) = 4$ which is finite. Hence, (3, 4) is a shaded circle on the graph of h(x). The limit of the function as $x \to 3$ is $\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 3 + 4 = 6$ (finite). The first two conditions of continuity of a function at a number are satisfied by the function, however, the third is not, that is, $\lim_{x \to 3} h(x) \neq g(3)$. Hence, we conclude that the function $h(x)$ is not continuous or is discontinuous at $x = 3$ and this property of the function is seen on the graph of $h(x)$.

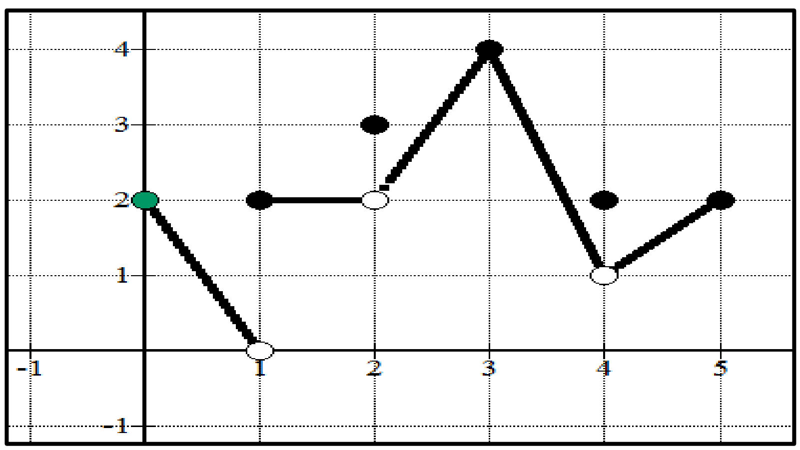


c). For piecewise-defined function $k(x) = \begin{cases} \frac{1}{x}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

The function value k (1) = 0 . The $\lim_{x\to 1^+} k(x) = 1$ and $\lim_{x\to 1^-} k(x) = 1$, therefore, the $\lim_{x\to 1} k(x) = 1$. We say that at $x = 1$, the function $k(x)$ is discontinuous since $\lim_{x\to 1} k(x) \neq k(1)$ and this property of the function is revealed by its graph.



Example 7. Given the graph of function $g(x)$ observe the following properties of the function.



a). $g(1) = 2$	h). $\lim_{x\to 3^-} g(x) = 4$	o). $\lim_{x\to 4^-} g(x) = 1$
b). $g(2) = 3$	i). $\lim_{x\to 3^+} g(x) = 4$	p). $\lim_{x\to 4} g(x) = 1$
c). $g(3) = 4$	j). $\lim_{x\to 3} g(x) = 4$	q). Discontinuous at $x = 1$
d). $g(4) = 2$	k). $\lim_{x\to 2^-} g(x) = 2$	r). Discontinuous at $x = 2$
e). $\lim_{x\to 1^-} g(x) = 0$	l). $\lim_{x\to 2^+} g(x) = 2$	s). Discontinuous at $x = 4$
f). $\lim_{x\to 1^+} g(x) = 2$	m). $\lim_{x\to 2} g(x) = 2$	t). Continuous at $x = 3$
g). $\lim_{x\to 1} g(x) = \text{DNE}$	n). $\lim_{x\to 4^+} g(x) = 1$	

1.12. Continuity of a Function on an Interval

The following definitions facilitate explaining continuity of a function on an interval.

a) A function is continuous on an open interval (a,b) if an only if it is continuous at every number in the open interval.

b) A function $f(x)$ is said to be continuous at a from the right if and only if the following conditions are satisfied.

(i). $f(a)$ exists;

(ii). $\lim_{x \to a^+} f(x)$ exists; and,

(iii). $f(a) = \lim_{x \to a^+} f(x)$.

This continuity is called right-hand continuity.

c) A function $f(x)$ is said to be continuous at a from the left if and only if the following conditions are satisfied.

(i). $f(a)$ exists;

(ii). $\lim_{x \to a^-} f(x)$ exists; and,

(iii). $f(a) = \lim_{x \to a^-} f(x)$.

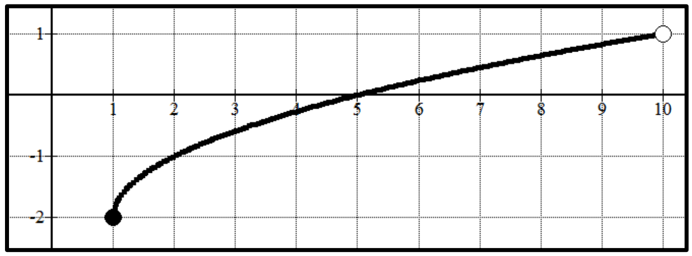
This continuity is called left-hand continuity.

d) A function $f(x)$ whose domain is the closed interval $[a,b]$ is said to be continuous at $[a,b]$ if and only if it is continuous on the open interval (a,b), as well as continuous from the right at a and continuous from the left at b. That is, $f(a) = \lim_{x \to a^+} f(x)$, and, $f(b) = \lim_{x \to b^-} f(x)$.

e) A function $f(x)$ whose domain includes the right open interval $[a, b)$ is said to be continuous at $[a, b)$ if it is continuous in the open interval (a, b) and continuous from the right at a, that is $f(a) = \lim_{x \to a^+} f(x)$.

f) A function whose domain includes the left open interval $(a, b]$ is said to be continuous at $(a, b]$ if it is continuous in the open interval (a, b) and continuous from the left at b, that is, $f(b) = \lim_{x \to b^-} f(x)$.

Example 8. Let us consider the sketch of the graph of $h(x)$ and investigate if it is continuous in the right open interval $[1, 10)$.

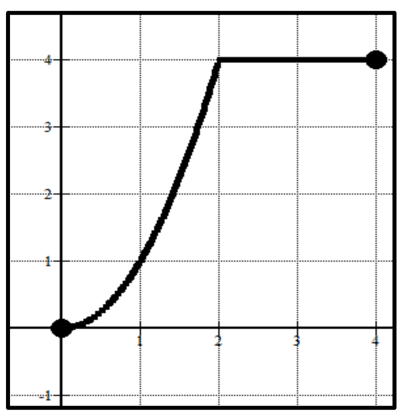


As seen from the graph, the condition that the function h must be continuous at any x-value in the right open interval$(1, 10)$ is satisfied. Now, examine if it is continuous from the right of $x = 1$? Are the three conditions for the right-hand continuity to exist satisfied?

- $h(1) = -2$
- $\lim_{x \to 1^+} h(x) = -2$
- $\lim_{x \to 1^+} h(x) = h(1) = -2$

From the above results, we conclude that the function h is continuous in the right open interval $[1, 10)$.

Example 9. Consider the sketch of the graph of function $f(x)$ and examine its continuity on the closed interval $[0.4]$.



As reflected on the graph of function $f(x)$ whose domain is the closed interval$[0, 4]$, it is continuous at any $x -$ value on the open interval $(0,4)$. Moreover, observe that $f(0) = \lim_{x \to 0^+} f(x) = 0$. Hence, the function is continuous at $x = 0$.

Furthermore, $f(4)=\lim_{x\to 4^-} f(x) = 4$. Therefore, we say that the function is continuous at $x=4$. Examining the continuity at $x=2$, note that $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2} f(x) = 4$ and $f(2)=4$. Hence, the function is said to be continuous at $x=2$.

Thus, we finally say that the function is continuous on the closed interval $[0.4]$

1.13. Intermediate Value Theorem

The Intermediate Value Theorem says that if function $f(x)$ is continuous on the closed interval $[a,b]$ such that there exists a number M between $f(a)$ and $f(b)$, then, there exists a number c such that $a<c<b$. All that is being said by the Intermediate Value Theorem is that a continuous function will take on all values between $f(a)$ and $f(b)$. In other words somewhere between a and b, the function will take on the value of *M.*

Example 10. Show that $f(x)=2x^3-5x^2-10x+5$ has a root somewhere in the interval [-1, 2]

Solutions: What we're really asking here is whether or not the function will take on the value $f(x)=0$ somewhere between -1 and 2. In other words, we want to show that there is a number c such that $-1<c<2$ and $f(x)=0$. However if we define $M=0$ and acknowledge that $a=-1$ and $b=2$ we can see that these two condition on c are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M=0$ is between $f(-1)$ and $f(2)$. To do this all we need to do is compute, $f(-1)=2x^3-5x^2-10x+5$ and $f(2)=2x^3-5x^2-10x+5$

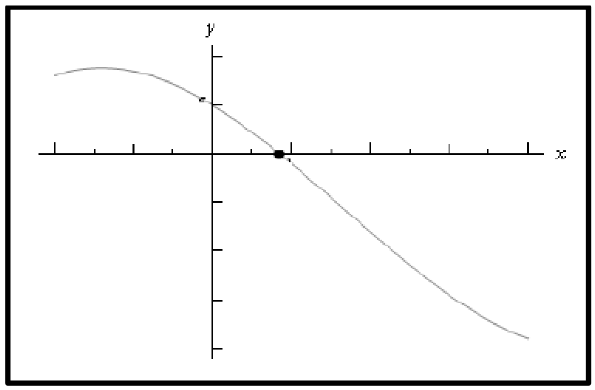
$f(-1)=2x^3-5x^2-10x+5$ $\qquad$ $f(2)=2x^3-5x^2-10x+5$

$f(-1) = 2(-1)^3 - 5(-1)^2 - 10(-1) + 5$
$f(-1) = 2(-1) - 5(1) + 10 + 5$
$f(-1) = -2 - 5 + 10 + 5$
$f(-1) = -7 + 15$
$f(-1) = 8$

$f(2) = 2(2)^3 - 5(2)^2 - 10(2) + 5$
$f(2) = 2(8) - 5(4) - 10(2) + 5$
$f(2) = 16 - 20 - 20 + 5$
$f(2) = -4 - 15$
$f(2) = -19$

So we have, $-19 = f(2) < 0 < f(-1) = 8$

Therefore, $M = 0$ is between $f(-1)$ and $f(2)$, since $f(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that $f(c) = 0$. Therefore the polynomial does have a root between -1 and 2. The graph is shown below.

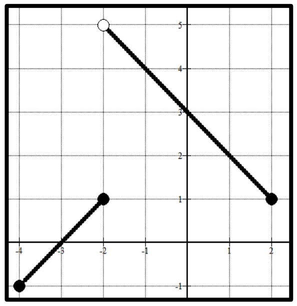


1.14. Types of Discontinuity

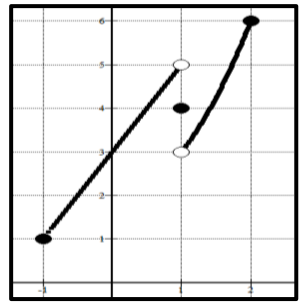
Seeing how a function fails to become continuous helps one to understand discontinuity. All of the important functions used in Calculus and analysis are continuous except at isolated points. Such points are called *points of discontinuity*. To better understand the succeeding discussions on discontinuity requires one to remember the three conditions satisfied by a continuous function. That is, function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a) =$ defined. Thus, if $x = a$ is a point of discontinuity, something about the above condition fails to be true. Discontinuity of a function may be of the following types:

A. Jump Discontinuity

In a jump discontinuity, the right-hand and the left-hand limits both exist as $x \to a$, but are not equal. Thus, the $\lim_{x \to a} f(x)$ does not exist. The size of the jump is the difference between the right-hand and the left-hand limits. The graphs below show jump discontinuity.



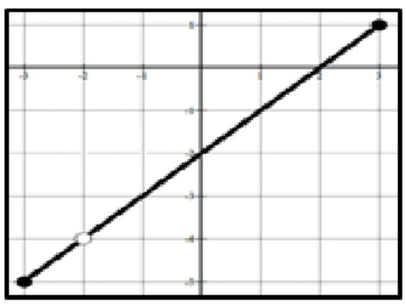
Point of discontinuity: $x = -2$



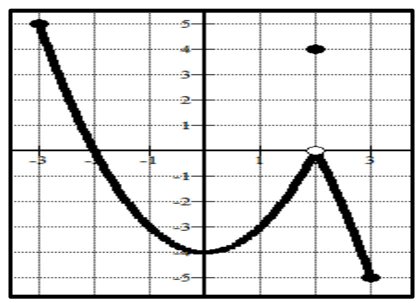
Point of discontinuity: $x = 1$

B. Hole/Removable Discontinuity

In a removable discontinuity, the $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$. This may be because $f(a)$is undefined or indeterminate. The discontinuity can be removed by changing the definition of $f(x)$ at a so that its new value, there is $\lim_{x \to a} f(x)$. Consider the graphs below.



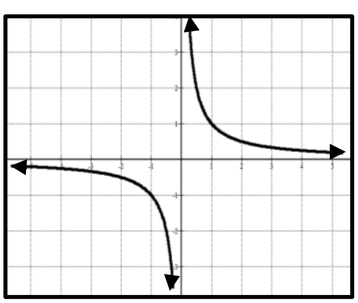
Point of discontinuity: $x = -2$



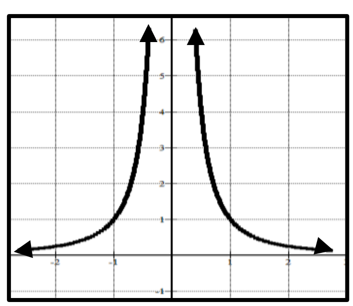
Point of discontinuity: $x = 2$

C. Asymptotic/Infinite Discontinuity

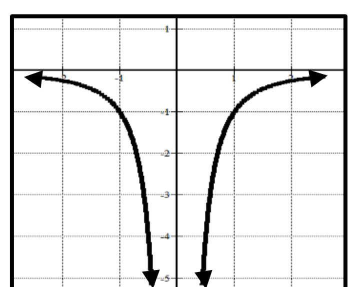
In an infinite discontinuity, the left- and right-hand limits are infinite; they may be both positive, both negative, or one positive and one negative. Since the function doesn't approach a particular finite value, the limit does not exist. The graphs given on the next page are all discontinuous at $x = 0$.



$\lim_{x \to 0^-} f(x) = -\infty$;
$\lim_{x \to 0^+} f(x) = \infty$



$\lim_{x \to 0^-} f(x) = \infty$;
$\lim_{x \to 0^+} f(x) = \infty$



$\lim_{x \to 0^-} f(x) = -\infty$;
$\lim_{x \to 0^+} f(x) = -\infty$

Asymptotic/infinite discontinuity occurs at points where the denominator of a rational function equal to zero. So all that we need to determine this kind of discontinuity is to set the denominator to zero and solve for x.

For example, function $f(x) = \frac{2x-3}{x^2+5x-6}$ has infinite discontinuity at $x = -6$ and $x = 1$since at these values, the denominator of the rational function is zero.

Activity Sheet
LIMIT OF A FUNCTION

Name : ______________________________ **SCORE :** __________

Section :_______________________**Class Schedule :** ________________

Note: Students may use calculators when applicable.

1. Complete the following tables of values to investigate $\lim_{x \to 0} \frac{x-1}{x+1}$

x	f(x)
−1	
−0.8	
−0.35	
−0.1	
−0.09	
−0.0003	
−0.000001	

x	f(x)
1	
0.75	
0.45	
0.2	
0.09	
0.0003	
0.000001	

2. Complete the following tables of values to investigate $\lim_{x \to 1} \left(x^2 - 2x + 4\right)$

x	f(x)
0.5	
0.7	
0.95	
0.995	
0.9995	
0.99995	

x	f(x)
1.6	
1.35	
1.05	
1.005	
1.0005	
1.00005	

3. Construct a table of values to investigate the following limits:

a. $\lim_{x\to 1} \frac{1}{x+1}$

b. $\lim_{x\to 0} f(x)$ if $f(x) = \begin{cases} \frac{1}{x} & , if\ x \le -1 \\ x^2 - 2, & if\ x > -1 \end{cases}$

Activity Sheet
LIMIT THEOREM

Name : ______________________________ **SCORE :** ____________

Section :_____________________**Class Schedule :** ______________________

Directions: Evaluate the indicated limits using direct substitution method.

1. $\lim_{x \to -1} (2x)$

2. $\lim_{x \to 9} (x - 6)$

3. $\lim_{x \to 0} (x^2 - 5x + 9)$

4. $\lim_{x \to -2} (x^5 - 3x^3 + 4x^2 - 6)$

5. $\lim_{x \to -5} 4(x - 4)(x + 9)$

6. $\lim_{x \to 3} (4x - 4)^3$

7. $\lim_{x \to \frac{2}{5}} \frac{5x-7}{15x+2}$

8. $\lim_{x \to -2} \frac{2x^2-7}{3x^3+8}$

9. $\lim_{x \to 16} \sqrt[3]{4x}$

10. $\lim_{x \to 1} \sqrt{\frac{5x+4}{3x+1}}$

Activity Sheet
LIMIT THEOREM

Name : ______________________________ **SCORE :** ____________

Section :______________________________**Class Schedule :** ______________________

Evaluate the indicated limits.

1. $\lim_{x \to -5} \frac{x+5}{x^2+7x+10}$

2. $\lim_{x \to 2} \frac{x^2-4}{x-2}$

3. $\lim_{w \to 4} \frac{3w^2-8w-16}{2w^2-9w+4}$

4. $\lim_{v \to -3} \frac{v^3+27}{v+3}$

5. $\lim_{b \to 0} \frac{(3+b)^2-9}{b}$

6. $\lim_{c \to -2} \frac{c+2}{c^3+8}$

7. $\lim_{k\to 4} \frac{4-k}{2-\sqrt{k}}$

8. $\lim_{m\to 4} \frac{4m-m^2}{2-\sqrt{m}}$

9. $\lim_{a\to 2} \frac{\sqrt{a^2+12}-4}{a-2}$

10. $\lim_{d\to 1} \frac{d-1}{\sqrt{d+3}-2}$

11. $\lim_{x\to\infty} \frac{x^2}{1-x^2}$

12. $\lim_{x\to+\infty} \frac{2x+3}{3x+5}$

13. $\lim_{x\to+\infty} \frac{x^4+2x^2+5}{2x^3-6x+1}$

14. $\lim_{x\to+\infty} \frac{\sqrt{x^2+4}}{x+1}$

Activity Sheet
LIMIT OF TRANSCENDENTAL FUNCTION

Name : ____ **SCORE :** ____________

Section :________________________________**Class Schedule :** ________________________

Directions: Evaluate the following limits by constructing the table of values.

1. $\lim_{x \to 1} 4^x$

2. $\lim_{x \to 3} 5^{x^2 - 2x + 1}$

3. $\lim_{x \to 0} \cos x$

4. $\lim_{t \to 0} \frac{\sin 4t}{t}$

5. $\lim_{x \to 0} \frac{\cos x}{\sin x - 3}$

Activity Sheet
CONTINUITY OF A FUNCTION

Name : ______________________________ **SCORE :** __________

Section :______________________**Class Schedule :** ______________

Directions: Identify the type of discontinuity and use the definition of continuity to determine whether f is continuous at the indicated value of x.

1. $f(x) = 3x + 1$; x = 3

2. $f(x) = 4x^2 + 7x + 2$; x = 2

3. $f(x) = \frac{x^2 + 4}{x - 2}$; x = 4

4. $f(x) = \frac{x^2 + 7}{x - 3}$; x = 5

5. $f(x) = \frac{x + 5}{x - 5}$; x = 5

6. $f(x) = \dfrac{x-7}{x+7}$; x = 7

7. $f(x) = \dfrac{x^2+4x}{x^2-4x}$; x = 0

8. $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 4 \\ & x = 4 \\ 1, & x = 4 \end{cases}$

9. $f(x) = \begin{cases} 2-x, & x < 0 \\ x^2+1, & x \geq 0 \end{cases}$; x = 0

10. $f(x) = \begin{cases} x^2+2x-3, & x \leq -1 \\ x-3, & x > -1 \end{cases}$; x =−1

Activity Sheet
PROPERTIES OF CONTINUITY

Name : ______________________________ **SCORE :** __________

Section :____________________ **Class Schedule :** ______________

A. Using the properties of continuity, determine where each function is continuous.

1. $f(x) = 4x + 7$
2. $f(x) = 8x^2 - 5x + 2$
3. $f(x) = \frac{3x}{x+2}$
4. $f(x) = \frac{x+8}{x+1}$
5. $f(x) = \frac{x+5}{(x+8)(x-3)}$
6. $f(x) = \frac{x-4}{4x^2-9}$
7. $f(x) = \frac{x^2+8x}{x^2-8x}$
8. $f(x) = \sqrt{27-x}$
9. $f(x) = \sqrt{x^2-9}$
10. $f(x) = \sqrt[3]{x-9}$

B. Determine where the given function is continuous using the composition of continuous functions property.

1. $f(x) = \sqrt{\frac{x-5}{x}}$
2. $f(x) = \sqrt{\frac{x}{x^2-16}}$
3. $f(x) = \sin\left(x^2-4\right)$
4. $f(x) = \cos\sqrt[3]{8-x}$
5. $f(x) = \tan\left(\frac{7}{x^2+7}\right)$

Activity Sheet
INTERMEDIATE VALUE THEOREM

Name : ______________________________ **SCORE :** __________

Section :______________________________**Class Schedule :** ____________________

A. Use the Intermediate Value Theorem to verify that the function has a root in the given interval.

1. $f(x) = x^2 - 3, \ [-1,3]$
2. $f(x) = x^3 - 7x - 3, \ [3,2]$
3. $f(x) = x^3 - 7x - 3, \ [-1,1]$
4. $f(x) = x^3 - 7x - 3, \ [-2,-3]$
5. $f(x) = \cos x + x, \ [-2,0]$

B. Use the intermediate Value Theorem to show that the two functions intersect in the given interval.

1. $f(x) = x^2 + 1, \quad g(x) = x + 1, \ [-1,2]$
2. $f(x) = 5x^3 + 4x - 1, \quad g(x) = 2x^2 - 4x, \ [-2,0]$
3. $f(x) = \dfrac{1}{x^2}, \quad g(x) = -x^2 + 5, \ [2,1]$
4. $f(x) = \ln x, \quad g(x) = e^x, \ [-1,2]$
5. $f(x) = \cos x, \quad g(x) = x - 1, \ [1,0]$

CHAPTER 2

DERIVATIVES OF ALGEBRAIC FUNCTIONS

Learning Competencies

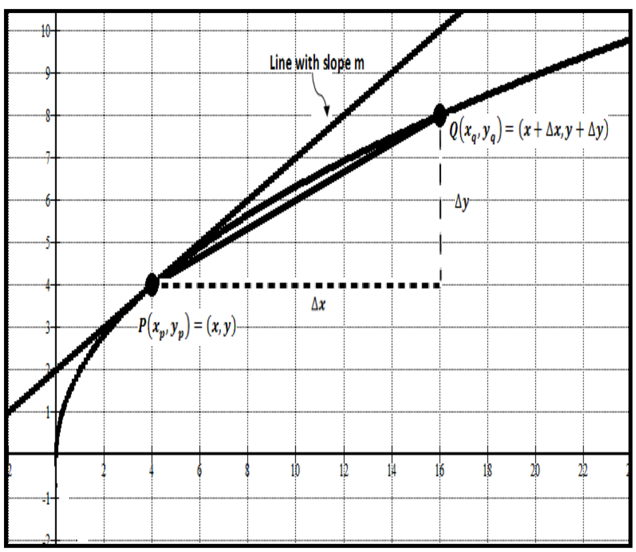
At the end of the chapter, the students learn to:

- Apply the definition of derivative of a function at a given number using increment.
- Illustrate the tangent line to the graph of a function at a given point.
- Relate the derivative of a function to the slope of the tangent line.
- Determine the relationship between differentiability and continuity of a function.
- Apply the differentiation rules in computing the derivative of algebraic, exponential, logarithmic, trigonometric and inverse trigonometric functions.
- Compute the higher-order derivatives.
- Illustrate the Chain Rule of differentiation.
- Solve problems using Chain Rule of differentiation.
- Illustrate implicit differentiation.
- Solve problems using implicit differentiation.
- Solve optimization problems.
- Solve situational problems involving related rates.

2.1. Increment

An increment is a small, unspecified, nonzero change in the value of a quantity. The symbol most commonly used is the uppercase Greek letter delta (Δ).

Consider the case of the graph of a function $y = f(x)$ in Cartesian coordinates, as shown in the figure. The slope of this curve at a specific point P is defined as the limit of $\frac{\Delta y}{\Delta x}$ as Δx (read "delta x") approaches zero, provided the function is continuous. That is, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The value $\frac{\Delta y}{\Delta x}$ depends on defining two points in the vicinity of P. In the illustration, one of the points is P itself, defined as (x_P, y_P) and the other is $Q(x_q, y_q)$, which is near P. The increments here are $\Delta y = y_q - y_p$ and $\Delta x = x_q - x_p$. As point Q approaches point P, both of these increments approach zero, and the ratio of increments $\frac{\Delta y}{\Delta x}$ approaches the slope of the curve at point P.



When the increment is positive, it means "increase in the value of the quantity" while a negative increment signifies a "decrease". The term increment is occasionally used in physics and engineering to represent a small change in a parameter such as temperature T (ΔT), electric current I (ΔI) or time t (Δt).

2.2. Derivative

We will extend our discussion of limits and examine the idea of the derivative, the basis of differential calculus. We will assume a particular function of x, such that $y = f(x) = x^2$

If x is assigned the value 5, the corresponding value of y will be $(5)^2$ or 25. Now, if we increase the value of x by 3, making the new x value 8, we have increment $\Delta x = 3$. This results in an increase in the corresponding value of y, and we call this increase an increment or Δy. From this we write

$$y + \Delta y = (x + \Delta x)^2 = (5 + 3)^2 = 64$$

Thus,

$$\Delta y = (x + \Delta x)^2 - y$$

$$\Delta y = (5 + \Delta x)^2 - 25 = 64 - 25 = 39$$

We are interested in the ratio $\frac{\Delta y}{\Delta x}$ because the limit of this ratio as Δx approaches zero is, by definition, the derivative of function f with respect to x.

As we recall from the discussion of limits, as Δx is made smaller, Δy gets smaller also. In our example, the ratio $\frac{\Delta y}{\Delta x}$ approaches 10 as shown on the table below. Let x = 5, correspondingly, y = 25, then assume values of Δx that tend to approach zero through values more than zero. Observe that as $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to 10$.

Variable	Δx	1	0.1	0.01	0.001	0.0001	0.00001
	$\Delta y = (x + \Delta x)^2 - y$ $\Delta y = (x + \Delta x)^2 - x^2$	11	1.01	0.1001	0.010001	1.00001 x10^{-3}	1.000001 x10^{-4}
	$\frac{\Delta y}{\Delta x}$	11	10.1	10.01	10.001	10.0001	10.00001

The symbol $\frac{\Delta y}{\Delta x}$ gives the average rate of change of y with respect to x, that is, with x changing from x to $x + \Delta x$, and with y correspondingly changing from y to $y + \Delta y$. In effect, the value of the function $f(x)$ becomes $y = f(x + \Delta x)$. Furthermore, if for a fixed value of x, the quotient $\frac{\Delta y}{\Delta x}$ approaches a limit as the increment Δx approaches zero, this limit is called the derivative of y with respect to x for the given value of x.

This is denoted by symbol $\frac{dy}{dx}$ or $\frac{d}{dx} f(x), y', f'(x), D_x y, D_x f(x)$.

Thus, by definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiation is the process of finding $\frac{dy}{dx}$ when $y = f(x)$. If the derivative of $f(x)$ exists, then, $f(x)$ is said to be a differentiable function of x.

Note that if $y = f(x)$, the instantaneous rate of change of y per unit change in x at x_1 is $f'(x_1)$, $y'(x_1)$ or, equivalently, the derivative of y with respect to x at x_1, if it exists.

2.3. The Increment Method/Four-Step Rule of Differentiation

This is the long method of finding the derivative of a given function using the increment of a variable and it may be formulated as follows:

1. Replace $(x + \Delta x)$ for x and $(y + \Delta y)$ for y.
2. To get Δy, subtract the original function of x, $f(x)$, from the new function of $(x + \Delta x)$, $f(x + \Delta x)$. Thus, $\Delta y = f(x + \Delta x) - f(x)$.
3. Divide both sides of the resulting equation in step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.
4. Take the limit as Δx approaches zero of all the terms in the equation from Step 3. The resulting equation is the derivative of f(x) with respect to x or simply $\frac{dy}{dx}$.

Example 1. Using the 4-Step Rule/Increment Method, find the derivative of y with respect to x or $\frac{dy}{dx}$ of the given functions when x = 2.

a). $y = x^2 + 2x - 3$

b). $y = (x - 1)(3x + 2)$

c). $y = \frac{1}{(x-1)^2}$

Solution:

a). $y = x^2 + 2x - 3$ ……………………………………………………………… (1)

Step 1: Replace y by $y + \Delta y$ and x by $x + \Delta x$ on the given function.

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 3 \quad \text{.................................... (2)}$$

Step 2: Subtracting (2) to (1) will give Δy.

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 3$$
$$-\,y = x^2 + 2x - 3$$
$$\overline{\Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 3 - (x^2 + 2x - 3)}$$

Expanding square of a binomial of the form $(a+b)^2 = a^2 + 2ab + b^2$, we arrive at

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x - 3 - x^2 - 2x + 3$$

Combining similar terms results to

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 2\Delta x$$

Factor-out Δx from the terms at the right of the above equation.

$$\Delta y = \Delta x(2x + \Delta x + 2)$$

Step 3: Divide the resulting equation in Step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 2$$

Step 4. Take the limit as x approaches zero of both sides of the above equation.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 0 + 2 = 2(x+1)$$

From the definition of derivative of a function, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Therefore, at any value of x, $\frac{dy}{dx} = 2(x+1)$.

b). $y = (x-1)(3x+2)$.. (1)

Step 1: Step 1: Replace y by $y + \Delta y$ and x by $x + \Delta x$ on the given function.

$$y + \Delta y = (x + \Delta x - 1)[3(x + \Delta x) + 2] \quad \text{.............................. (2)}$$

Step 2: Subtracting (2) to (1) will give Δy.

$$\begin{array}{l} y + \Delta y = (x + \Delta x - 1)[3(x + \Delta x) + 2] \\ -\ y = (x-1)(3x+2) \\ \hline \Delta y = (x + \Delta x - 1)[3(x + \Delta x) + 2] - (x-1)(3x+2) \end{array}$$

Expanding and combining similar terms results to

$$\Delta y = [3x^2 + 3x\Delta x - 3x + 3x\Delta x + 3(\Delta x)^2 - 3x + 2x + 2\Delta x - 2] - (3x^2 + x + 2)$$

$$\Delta y = 3x\Delta x + 3x\Delta x + 3(\Delta x)^2 - 3\Delta x + 2\Delta x$$

$$\Delta y = 6x\Delta x + 3(\Delta x)^2 - \Delta x$$

Factor-out Δx from the terms at the right of the above equation.

$$\Delta y = \Delta x(6x + 3\Delta x - 1)$$

Step 3: Divide the resulting equation in Step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x - 1$$

Step 4. Take the limit as x approaches zero of both sides of the above equation.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 1) = 6x + 0 - 1 = 6x - 1$$

From the definition of derivative of a function, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Therefore, at any value of x, $\frac{dy}{dx} = 6x - 1$.

c). $y = \frac{1}{(x-1)^2}$, when x = 2……………………………………………………... (1)

Step 1: Step 1: Replace y by $y + \Delta y$ and x by $x + \Delta x$ on the given function.

$$y + \Delta y = \frac{1}{(x + \Delta x - 1)^2} \quad \text{……………………………………..} \ (2)$$

Step 2: Subtracting (2) to (1) will give Δy.

$$\Delta y = \frac{1}{(x + \Delta x - 1)^2} - \frac{1}{(x-1)^2}$$

Subtract the fractions by getting their least common denominator (LCD).

$$\Delta y = \frac{(x-1)^2 - (x + \Delta x - 1)^2}{(x + \Delta x - 1)^2 (x-1)^2}$$

Recall how to expand square of a trinomial:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$\Delta y = \frac{(x^2 - 2x + 1) - \left[x^2 + (\Delta x)^2 + 1 + 2x\Delta x - 2x - 2\Delta x\right]}{(x + \Delta x - 1)^2 (x-1)^2}$$

$$\Delta y = \frac{-(\Delta x)^2 - 2x\Delta x + 2\Delta x}{(x + \Delta x - 1)^2 (x-1)^2}$$

Factor-out Δx from the terms at the right of the above equation.

$$\Delta y = \frac{\Delta x(-\Delta x - 2x + 2)}{(x + \Delta x - 1)^2 (x-1)^2}$$

Step 3: Divide the resulting equation in Step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x - 2x + 2}{(x + \Delta x - 1)^2 (x-1)^2}$$

Step 4. Take the limit as x approaches zero of both sides of the above equation.

$$\lim_{\Delta x \to 0} \frac{-\Delta x - 2x + 2}{(x + \Delta x - 1)^2 (x-1)^2}$$

From the definition of derivative of a function $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-\Delta x - 2x + 2}{(x + \Delta x - 1)^2 (x-1)^2}$

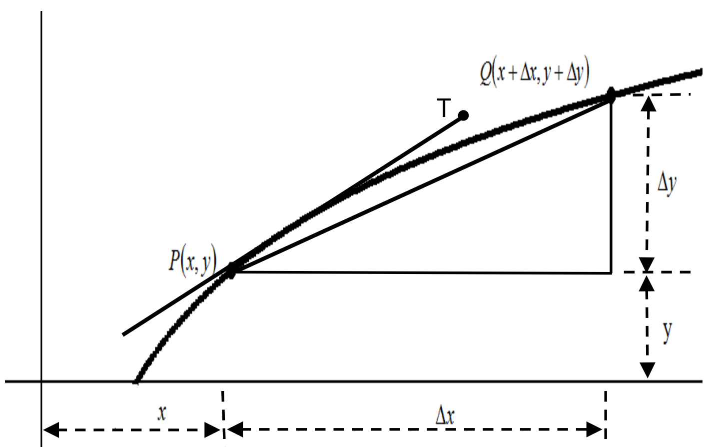
Therefore, $\frac{dy}{dx} = \frac{0 - 2x + 2}{(x + 0 - 1)^2 (x-1)^2} = \frac{-2x+2}{(x-1)^4} = \frac{-2(x-1)}{(x-1)^4} = \frac{-2}{(x-1)^3}$

The derivative above is true to all values of x, provided $x \neq 1$.

Therefore, at $x = 2$, $\frac{dy}{dx} = \frac{-2}{(2-1)^3} = \frac{-2}{(1)^3} = -2$

2.4. The Slope of the Tangent Line and the Derivative

Consider two distinct points $P(x, y)$, a fixed point, and a variable point $Q(x + \Delta x, y + \Delta y)$ on the graph of function $y = f(x)$. Line PQ is a secant while line PT the tangent line to the curve at point P.



Let point Q approach point P along the curve. From the figure, we see the slope of the secant line $PQ = \frac{QR}{PR} = \frac{\Delta y}{\Delta x}$. As $Q \to P$, that is, as $\Delta x \to 0$, the slope of PQ takes the slope of the tangent line at P as its limit. Thus, by definition, Slope of tangent line at $P(x,y) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ at point $P(x,y)$

Equation of tangent line at (x_1, y_1): $y - y_1 = \frac{dy}{dx}(x - x_1)$

The slope of the tangent line defines the slope of the curve at the point of tangency. Thus,

Slope of the curve $y = \frac{dy}{dx}$ at point (x_1, y_1)

Recall that a normal line to the curve of $y = f(x)$ is perpendicular to the tangent at the point of tangency (x_1, y_1). Therefore, the slope of the normal line is the negative reciprocal of the slope of the tangent.

In symbol form, slope of normal line at $(x_1, y_1) = -\frac{1}{\frac{dy}{dx}}$ at (x_1, y_1). Thus,

Slope of normal line at $P(x_1, y_1) = -\frac{1}{\frac{dy}{dx}}$ point $P(x_1, y_1)$

Equation of normal line at (x_1, y_1): $y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$

Example 2. Find the slope and equation of the tangent and normal line to the

a). parabola $y = 5x^2$ at point $(1,5)$.

b). hyperbola $y = \sqrt{x}$ at point $(4,2)$.

Solution:

a). parabola $y = 5x^2$ at point $(1,5)$

To find $\frac{dy}{dx}$, for the moment that we have not discussed the rules on differentiation, we use the increment method.

$$y + \Delta y = 5(x + \Delta x)^2$$

$$\Delta y = 5(x + \Delta x)^2 - 5x^2 = 5\left[x^2 + 2x\Delta x + (\Delta x)^2\right] - 5x^2$$

$$\Delta y = 5x^2 + 10x\Delta x + 5(\Delta x)^2 - 5x^2 = 10x\Delta x + 5(\Delta x)^2$$

$$\Delta y = 5\Delta x(2x + \Delta x)$$

$$\frac{\Delta y}{\Delta x} = 5(2x + \Delta x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 5(2x + \Delta x) = 5(2x + 0) = 10x$$

At point $(1,5)$, $\frac{dy}{dx} = 10(1) = 10$.

Therefore, the slope of the tangent line to curve $y = 5x^2$, which is given by $\frac{dy}{dx}$ is 10. We use the point-slope form of the equation of a line to find equation of the tangent line at point $(1,5)$. That is, $y - y_1 = m(x - x_1)$ where $m = \frac{dy}{dx} = 10$ and the point is $(1,5)$.

$$y - 5 = 10(x - 1)$$
$$y - 5 = 10x - 10$$
$$10x - y - 5 = 0$$

Hence, equation of the tangent line at $(1,5)$ is $10x - y - 5 = 0$.

The slope of the normal line $(1,5)$ is equal to $-\frac{1}{\frac{dy}{dx}} = -\frac{1}{10}$ and its equation is

$$y - 5 = -\frac{1}{10}(x - 1)$$
$$10y - 50 = -x + 1$$
$$x + 10y - 51 = 0$$

b). hyperbola $y = \sqrt{x}$ at point $(4,2)$

We replace y by $y + \Delta y$ and x by $x + \Delta x$.

$$y + \Delta y = \sqrt{x + \Delta x}$$
$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

To arrive at an equation having Δx as common factor of the terms at the right side of it, we multiply and divide the right side by $\sqrt{x + \Delta x} + \sqrt{x}$ which is the conjugate of $\sqrt{x + \Delta x} - \sqrt{x}$. That is,

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Simplifying the right side, $\Delta y = \frac{\left(\sqrt{x + \Delta x} - \sqrt{x}\right)\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}{\sqrt{x + \Delta x} + \sqrt{x}}$

Observe that the numerator of the fraction is the product of the sum and difference of two terms which when multiplied results to a difference of two squares. That is, $(a + b)(a - b) = a^2 - b^2$. Hence,

$$\Delta y = \frac{\left(\sqrt{x+\Delta x}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\Delta y = \frac{(x+\Delta x) - x}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\Delta y = \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At point $(4,2)$, slope of the tangent to the hyperbola is equal to $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

To the hyperbola at point $(4,2)$, equation of the tangent line is

$$y - 2 = \frac{1}{4}(x-4)$$

$$4y - 8 = x - 4$$

$$x - 4y + 4 = 0$$

At point $(4,2)$, slope of the normal to the hyperbola is equal to $-\frac{1}{\frac{dy}{dx}} = -\frac{1}{\frac{1}{4}} = -4$ and

its equation is:

$$y - 2 = -4(x-4)$$

$$y - 2 = -4x + 16$$

$$4x + y - 18 = 0$$

2.5. Derivatives of Algebraic Functions

An algebraic function is one formed by a finite number of algebraic operations on constants and/or variables. These algebraic operations include addition, subtraction, multiplication, division, raising to powers, and extracting roots.

Description		Differentiation Rule/Formula
1. Derivative of a constant	:	$\frac{d}{dx}(c) = 0$
2. Derivative of a variable x with respect to x	:	$\frac{d}{dx}(x) = 1$
3. Derivative of power of variable x with respect to x	:	$\frac{d}{dx}(cx^n) = cnx^{n-1}$

4. Derivative of square root of variable x with respect to x	:	$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$
5. Derivative of ratio of a constant to variable x	:	$\frac{d}{dx}\left(\frac{c}{x}\right)=\frac{-c}{x^2}$
6. Derivative of a sum/difference of functions of a variable	:	$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$

The following differentiation formulas were derived using the 4-Step Rule of differentiation. They facilitate finding derivative of a function. On the given table of differentiation rule/formula, x is a variable; c and n are constants.

Example 3. Using the appropriate differentiation rule, find $\frac{dy}{dx}$.

a). $y=x^3-4x^2+6x-8$

First, differentiate both sides of the given equation using Rule 6, that is, $\frac{d}{dx}\left[f(x)\pm g(x)=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)\right]$. Therefore,

$$\frac{dy}{dx}=\frac{d}{dx}(x^3)-\frac{d}{dx}(4x^2)+\frac{d}{dx}(6x)+\frac{d}{dx}(-8)$$

Then, apply $\frac{d}{dx}(cx^n)=cnx^{n-1}$ and $\frac{d}{dx}(c)=0$.

$$\frac{dy}{dx}=3x^{3-1}-4(2)x^{2-1}+6\left(\frac{dx}{dx}\right)+(0)$$

$$\frac{dy}{dx}=3x^2-8x+6(1)=3x^2-8x+6$$

b). $y=3x^3-6x+\frac{4}{x}$

$$\frac{dy}{dx}=\frac{d}{dx}(3x^3)-\frac{d}{dx}(6x)+\frac{d}{dx}\left(\frac{4}{x}\right)$$

$$\frac{dy}{dx}=3(3x^2)-6\left(\frac{dx}{dx}\right)+\left(\frac{-4}{x^2}\right)$$

$$\frac{dy}{dx}=9x^2-6(1)-\frac{4}{x^2}$$

$$\frac{dy}{dx}=9x^2-6-\frac{4}{x^2}$$

c). $y = \frac{3}{x} - 4x + 4\sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{x}\right) - 4\left(\frac{dx}{dx}\right) + 4\frac{d}{dx}\sqrt{x}$$

$$\frac{dy}{dx} = \frac{-3}{x^2} - 4(1) + 4\left(\frac{1}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = -\frac{3}{x^2} - 4 + \frac{2}{\sqrt{x}}$$

d). $y = (2x^2 - 3)^2$

First, we expand the right side of the given equation to bring it into a sum of terms. This is done by using the special product square of a binomial, that is, $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Hence,

$$y = (2x^2 - 3)^2$$

$$y = 4x^4 - 12x^2 + 9$$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^4) - \frac{d}{dx}(12x^2) + \frac{d}{dx}(9)$$

$$\frac{dy}{dx} = 4(4x^{4-1}) - 12(2x^{2-1}) + 0$$

$$\frac{dy}{dx} = 16x^3 - 24x = 8x(2x^2 - 3)$$

e). $y = \frac{(4x-1)^3}{x^2}$

First, we expand the numerator to reduce the given ratio into a sum of terms by using the special product cube of a binomial $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 - b^3$. Then, divide the result by x^2. Hence, we arrive at

$$y = \frac{(4x)^3 - 3(4x)^2(1) + 3(4x)(1)^2 - (1)^3}{x^2}$$

$$y = \frac{64x^3 - 48x^2 + 12x - 1}{x^2}$$

$$y = \frac{64x^3}{x^2} - \frac{48x^2}{x^2} + \frac{12x}{x^2} - \frac{1}{x^2}$$

$$y = 64x - 48 + \frac{12}{x} - x^{-2}$$

Therefore,
$$\frac{dy}{dx}=\frac{d}{dx}(64x)-\frac{d}{dx}(48)+\frac{d}{dx}\left(\frac{12}{x}\right)-\frac{d}{dx}\left(x^{-2}\right)$$
$$\frac{dy}{dx}=64\left(\frac{dx}{dx}\right)-0+\left(\frac{-12}{x^2}\right)-\left(-2x^{-2-1}\right)$$
$$\frac{dy}{dx}=64(1)-\frac{12}{x^2}+2x^{-3}=64-\frac{12}{x^2}+\frac{2}{x^3}$$
$$\frac{dy}{dx}=\frac{64x^3-12x+2}{x^3}=\frac{2\left(32x^3-6x+1\right)}{x^3}$$

f). $y=\left(3x^4-2x^2+4x-1\right)\left(x^5-4x+2\right)$

Get the product of the two factors at the right side of the equation to bring it into a sum of terms. Doing so will result to
$$y=3x^9-2x^7+4x^6-13x^5+6x^4+8x^3-20x^2+12x-2$$
$$\frac{dy}{dx}=\frac{d}{dx}\left(3x^9\right)-\frac{d}{dx}\left(2x^7\right)+\frac{d}{dx}\left(4x^6\right)-\frac{d}{dx}\left(-13x^5\right)+\frac{d}{dx}\left(6x^4\right)+\frac{d}{dy}\left(8x^3\right)-\frac{d}{dx}\left(20x^2\right)+\frac{d}{dx}(12x)-\frac{d}{dx}(2)$$
$$\frac{dy}{dx}=3\left(9x^8\right)-2\left(7x^6\right)+4\left(6x^5\right)-13\left(5x^4\right)+6\left(4x^3\right)+8\left(3x^2\right)-20(2x)+12(1)-0$$
$$\frac{dy}{dx}=27x^8-14x^6+24x^5-65x^4+24x^3+24x^2-40x+12$$

Example 4. Find the slope of the given curve at the indicated point using the appropriate differentiation rules.

a). $y=\left(4x^2+3\right)^2;\ \left(\frac{1}{2},16\right)$

Solution:
First, we expand the right side of the given equation as square of a binomial.
$$y=\left(4x^2\right)^2+2\left(4x^2\right)(3)+(3)^2$$
$$y=16x^4+24x^2+9$$

Differentiating results to $\frac{dy}{dx}=16\left(4x^3\right)+24(2x)+0=64x^3+48x$

At point $\left(\frac{1}{2},16\right)$, $\quad \frac{dy}{dx}=64\left(\frac{1}{2}\right)^3+48\left(\frac{1}{2}\right)=64\left(\frac{1}{8}\right)+24=8+24=32$

Since the slope of the curve at the point of tangency is defined by the slope of the tangent line at that point, therefore, slope of the curve at $\left(\frac{1}{2},16\right)$ equals 32.

b). $y = \sqrt{x} + \frac{8}{x}$; $(4,4)$

Solution:

Differentiating using the appropriate differentiation rules yield

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \left(\frac{-8}{x^2}\right)$$

At point $(4,4)$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} - \frac{8}{(4)^2} = \frac{1}{2(2)} - \frac{8}{16} = \frac{1}{4} - \frac{1}{2} = \frac{1-2}{4} = -\frac{1}{2}$

Therefore, slope of the curve which is defined by the slope of the tangent line at $(4,4)$ is equal to $-\frac{1}{2}$.

There are instances that we need to differentiate power, product, quotient and the like involving polynomial functions of a variable. Listed next are the differentiation rules where u, v and w are functions of variable x; c and n are constants.

Description		**Differentiation Rule**
1. General Power Formula	:	$\frac{d}{dx}(cu^n) = cnu^{n-1}\frac{du}{dx}$
2. Special Power Formula	:	$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{d}{dx}(u)$
3. Product Formula	:	$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$
4. Special Product Formula	:	$\frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$
5. Quotient Formula	:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$
6. Special Quotient Formula	:	$\frac{d}{dx}\left(\frac{c}{u}\right) = \frac{-c}{u^2}\frac{d}{dx}(u)$

Observe how the differentiation rules are used on the following examples.

Example 5. Find the derivative of the following functions using the appropriate differentiation rule.

a). $y = 4(2x^2 - 3)^2$

Recall that this was already differentiated during our previous discussion. But this is done by expanding the given square of a binomial, then followed by the use of the formula on derivative of a sum/difference.

Now, the differentiation process is made a little shorter through the use of the more general power formula $\frac{d}{dx}\left(cu^{n}\right)=cnu^{n-1}\frac{du}{dx}$; where:

$$u = 2x^2 - 3,\ n = 2,\ \ c = 4$$

$$\frac{du}{dx} = 2(2x) = 4x$$

Substitution on the formula results to

$$\frac{dy}{dx} = 4(2)\left(2x^2 - 3\right)(4x)$$

$$\frac{dy}{dx} = 32x\left(2x^2 - 3\right)$$

b). $y = \left(3x^4 - 2x^2 + 4x - 1\right)\left(x^5 - 4x + 2\right)$

Here, the differentiation process requires the use of the product rule $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$. This is to avoid the task of bringing the product of the two factors into a sum of terms. Hence, where:

$$u = 3x^4 - 2x^2 + 4x - 1 \qquad v = x^5 - 4x + 2$$

$$\frac{du}{dx} = 12x^3 - 4x + 4 \qquad \frac{dv}{dx} = 5x^4 - 4$$

Substitution on the differentiation formula yields

$$\frac{dy}{dx} = \left(3x^4 - 2x^2 + 4x - 1\right)\left(5x^4 - 4\right) + \left(x^5 - 4x + 2\right)\left(12x^3 - 4x + 4\right)$$

$$\frac{dy}{dx} = 15x^8 - 10x^6 + 20x^5 - 5x^4 - 12x^4 + 8x^2 - 16x + 4 + \left(12x^8 - 48x^4 + 24x^3 - 4x^6 + 16x^2 - 8x + 4x^5 - 16x + 8\right)$$

$$\frac{dy}{dx} = 27x^8 - 14x^6 + 24x^5 - 65x^4 + 24x^3 + 24x^2 - 40x + 12$$

c). $y=\dfrac{1-2x}{1+2x}$

We use here the quotient rule $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{d}{dx}(u)-u\dfrac{d}{dx}(v)}{v^2}$, where:

$$u=1-2x \qquad v=1+2x$$

$$\frac{du}{dx}=-2 \qquad \frac{dv}{dx}=2$$

Substitution on the quotient formula results to

$$\frac{dy}{dx}=\frac{(1+2x)(-2)-(1-2x)(2)}{(1+2x)^2}$$

$$\frac{dy}{dx}=\frac{2[-(1+2x)-(1-2x)]}{(1+2x)^2}$$

$$\frac{dy}{dx}=\frac{2[-1-2x-1+2x]}{(1+2x)^2}=\frac{2[-2]}{(1+2x)^2}$$

$$\frac{dy}{dx}=\frac{-4}{(1+2x)^2}$$

d). $y=\sqrt{4-2x-3x^2}$

We apply the special power formula $\dfrac{d}{dx}\sqrt{u}=\dfrac{1}{2\sqrt{u}}\dfrac{d}{dx}(u)$, where:

$$u=4-2x-3x^2 \qquad \frac{du}{dx}=0-2-6x=-2(1+3x)$$

Therefore,

$$\frac{dy}{dx}=\frac{1}{2\sqrt{4-2x-3x^2}}[-2(1+3x)]$$

$$\frac{dy}{dx}=-\frac{1+3x}{\sqrt{4-2x-3x^2}}$$

e). $y=\dfrac{3}{x^4+12x^2-6}$

We apply the special quotient formula $\dfrac{d}{dx}\left(\dfrac{c}{u}\right)=\dfrac{-c}{u^2}\dfrac{d}{dx}(u)$, where:

$$c=3,\ u=x^4+12x^2-6,\ \frac{du}{dx}=4x^3+24x=4x(x^2+6)$$

Hence,

$$\frac{dy}{dx}=-\frac{3}{(x^4+12x^2-6)^2}[4x(x^2+6)]$$

$$\frac{dy}{dx}=\frac{-12x(x^2+6)}{(x^4+12x^2-6)^2}$$

The previous example can also be differentiated using the quotient formula

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}, \text{ where: } \quad u=3 \qquad \frac{du}{dx}=0$$

$$v=x^4+12x^2-6 \qquad \frac{dv}{dx}=4x^3+24x$$

Applying the quotient formula yields

$$\frac{dy}{dx}=\frac{(x^4+12x^2-6)(0)-3(4x^3+24x)}{(x^4+12x^2-6)^2}$$

$$\frac{dy}{dx}=\frac{-3(4x^3+24x)}{(x^4+12x^2-6)^2}=\frac{-3(4x)(x^2+6)}{(x^4+12x^2-6)^2}$$

$$\frac{dy}{dx}=\frac{-12x(x^2+6)}{x^4+12x^2-6}$$

f). $y=\dfrac{3x-4}{\sqrt{x}}$

Use the quotient formula $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$, where:

$$u=3x-4 \qquad v=\sqrt{x}$$

$$\frac{du}{dx}=3 \qquad \frac{dv}{dx}=\frac{1}{2\sqrt{x}}$$

Applying the formula leads to

$$\frac{dy}{dx}=\frac{\sqrt{x}(3)-3\left(\frac{1}{2\sqrt{x}}\right)}{x}=\frac{\frac{3\sqrt{x}(2\sqrt{x})-3}{2\sqrt{x}}}{x}$$

$$\frac{dy}{dx}=\frac{6x-3}{2\sqrt{x}}\cdot\frac{1}{x}=\frac{6x-3}{2x\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{3(2x-1)}{2x\sqrt{x}}$$

g). $y=\dfrac{(1-x)(2-x^3)}{3x^2}$

Again, the quotient rule will work here to differentiate the given function of x, where: $u=(1-x)(2-x^3)$ $\qquad v=3x^2$

$$\frac{du}{dx}=(1-x)(-3x^2)+(2-x^3)(-1) \qquad \frac{dv}{dx}=6x$$

$$\frac{du}{dx} = -3x^2 + 3x^3 - 2 + x^3 = 4x^3 - 3x^2 - 2$$

Observe that $\frac{du}{dx}$ was obtained using the product formula. Substitution of the above on the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$ results to

$$\frac{dy}{dx} = \frac{3x^2(4x^3 - 3x^2 - 2) - (1-x)(2-x^3)(6x)}{(3x^2)^2}$$

$$\frac{dy}{dx} = \frac{12x^5 - 9x^4 - 6x^2 - 6x(2 - 2x - x^3 + x^4)}{9x^4}$$

$$\frac{dy}{dx} = \frac{12x^5 - 9x^4 - 6x^2 - 12x + 12x^2 + 6x^4 - 6x^5}{9x^4}$$

$$\frac{dy}{dx} = \frac{6x^5 - 3x^4 + 6x^2 - 12x}{9x^4}$$

$$\frac{dy}{dx} = \frac{3x(2x^4 - x^3 + 2x - 4)}{9x^4}$$

$$\frac{dy}{dx} = \frac{2x^4 - x^3 + 2x - 4}{3x^3}$$

h). $y = \frac{(4x-1)^3}{x^2}$

We use the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$, having

$u = (4x-1)^3$ $\qquad$ $v = x^2$

$\frac{du}{dx} = 3(4x-1)^2 \frac{d}{dx}(4x-1)$ $\qquad$ $\frac{dv}{dx} = 2x$

$$\frac{du}{dx} = 3(4x-1)^2(4)$$

$$\frac{du}{dx} = 12(4x-1)^2$$

Note that we got $\frac{du}{dx}$ using the general power formula. Substitution on the quotient formula gives $y'=\frac{x^2\left[12(4x-1)^2\right]-(4x-1)^3(2x)}{(x^2)^2}$

To simplify, bring-out the common factor from the terms on the numerator.

$$y'=\frac{2x(4x-1)^2[6x-(4x-1)]}{x^4}$$

$$y'=\frac{2x(4x-1)^2(6x-4x+1)}{x^4}$$

$$y'=\frac{2x(4x-1)^2(2x+1)}{x^4}$$

$$y'=\frac{2(4x-1)^2(2x+1)}{x^3}$$

Note: It is advised to always express, if ever possible, the derivative of a function in its factored-form.

i). $y=\left(\frac{x^3+8}{2x^3-1}\right)^4$

To differentiate the given function, first, we apply the power formula $\frac{d}{dx}(cu^n)=cnu^{n-1}\frac{du}{dx}$, where: $u=\frac{x^3+8}{2x^3-1}$ and $n=4$. However, to know $\frac{du}{dx}$, we use the quotient formula. That is,

$$\frac{du}{dx}=\frac{(2x^3-1)(3x^2)-(x^3+8)(6x^2)}{(2x^3-1)^2}$$

$$\frac{du}{dx}=\frac{(6x^5-3x^2)-(6x^5+48x^2)}{(2x^3-1)^2}=\frac{-51x^2}{(2x^3-1)^2}$$

Therefore, $\frac{dy}{dx}=4\left(\frac{x^3+8}{2x^3-1}\right)^3\left[\frac{-51x^2}{(2x^3-1)^2}\right]$

$$\frac{dy}{dx}=\frac{-204x^2(x^3+8)^3}{(2x^3-1)^3(2x^3-1)^2}$$

$$\frac{dy}{dx}=\frac{-204x^2(x^3+8)^3}{(2x^3-1)^5}$$

Example 6. Find the slope of the given curve at the indicated point.

a). $y = (4x^2+3)^2$; $\left(\frac{1}{2}, 16\right)$

Solution:

We use the power formula $\frac{d}{dx}(cu^n) = cnu^{n-1}\frac{d}{dx}(u)$ to find $f'(x) = \frac{dy}{dx}$.

$$f'(x) = \frac{dy}{dx} = 2(4x^2+3)(8x) = 16x(4x^2+3)$$

At point $\left(\frac{1}{2}, 16\right)$, slope of the tangent line is given by

$$\frac{dy}{dx} = f'\left(\frac{1}{2}\right) = 16\left(\frac{1}{2}\right)\left[4\left(\frac{1}{2}\right)^2 + 3\right] = 8(1+3) = 32$$

Since the slope of the curve at the point of tangency is defined by the slope of the tangent line at that point, therefore, slope of the curve at $\left(\frac{1}{2}, 16\right)$ equals 32.

b). $y = 2x^2\sqrt{4-x}$; $(0,0)$

Solution: Use the product formula of differentiation, $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

$$\frac{dy}{dx} = f'(x) = 2x^2\left[\frac{1}{2\sqrt{4-x}}(-1)\right] + \sqrt{4-x}(4x)$$

$$\frac{dy}{dx} = \frac{-x^2}{\sqrt{4-x}} + 4x\sqrt{4-x}$$

Simplify. $$\frac{dy}{dx} = \frac{-x^2 + 4x(4-x)}{\sqrt{4-x}}$$

$$\frac{dy}{dx} = \frac{-x^2 + 16x - 4x^2}{\sqrt{4-x}} = \frac{16x - 5x^2}{\sqrt{4-x}}$$

At point $(0,0)$, $f'(0) = \frac{dy}{dx} = \frac{0}{2} = 0$. Therefore, slope of the curve at $(0,0)$ which is equal to the slope of the tangent at $(0,0)$ is zero. This implies that the tangent to the curve at $(0,0)$is a horizontal line.

Example 7. Find the equation of the tangent and normal line to the graph of $y = \frac{4}{x+1}$ at $(1,2)$.

Solution: Using the differentiation formula $\frac{d}{dx}\left(\frac{c}{u}\right)=\frac{-c}{u^2}\frac{d}{dx}(u)$, we got $y'=\frac{-4}{(x+1)^2}(1)$.

At point (1.2), $\frac{dy}{dx}=\frac{-4}{(1+1)^2}=\frac{-4}{4}=-1$. Hence, slope of the tangent line at $(1,2)$ equals 1. Using the point-slope form of the equation of a line:

Equation of tangent line at $(1,2)$:

$$y-2=-1(x-1)$$
$$y-2=-x+1$$
$$x+y-3=0$$

The slope of the normal line at point (1.2) is equal to $-\frac{1}{\frac{dy}{dx}}=-\frac{1}{-1}=1$ and its equation is:

$$y-2=1(x-1)$$
$$y-2=x-1$$
$$x-y+1=0$$

Example 8. At what point is the tangent to the curve $y=\frac{16}{x}-x^2$ horizontal?

Solution:

Slope of a horizontal line is equal to zero. We, then find $\frac{dy}{dx}$ which is equal to the slope of the tangent line using the special quotient formula.

$$\frac{dy}{dx}=\frac{-16}{x^2}-2x$$

Replace $\frac{dy}{dx}$ by zero. $\quad 0=\frac{-16}{x^2}-2x$

Simplifying, $\quad \frac{16}{x^2}=-2x$

$$x^3=-8$$
$$x=\sqrt[3]{-8}$$
$$x=-2$$

Substitute $x=-2$ on the equation of the curve to find the corresponding value of y.

$$y=\frac{16}{-2}-(-2)^2=-8-4=-12$$

Hence, the tangent line to the given curve is horizontal at point $(-2,-12)$.

Example 9. At what points are the tangent to the curve $y = x^3 + 4$ parallel to line $y - 12x + 1 = 0$?

Solution:

Reducing the given equation of line $y - 12x + 1 = 0$ to slope-intercept $y = mx + b$ gives slope of line equal to 12. This is equal to the slope of the tangent line at the unknown points. Therefore,

$$\frac{dy}{dx} = 3x^2$$
$$12 = 3x^2$$
$$x^2 = 4$$
$$x = \pm 2$$

Substitute x-values on equation $y = x^3 + 4$ of the curve to get the corresponding y-values. When $x = 2, y = 12$ while when $x = -2, y = -4$. So, the points on the curve where the tangent is parallel to $y - 12x + 1 = 0$ are $(2, 12)$ and $(-2, -4)$.

Example 10. Find the equation of the tangent line to the curve $y = x^2 - x - 6$ at the points of intersection of the curve with the x-axis.

Solution:

First, find the point of intersection of the curve $y = x^2 - x - 6$ and the x-axis $y = 0$. Using substitution method,

$$0 = x^2 - x - 6 = (x+2)(x-3)$$

Solve for x, $x = -2$ and $x = 3$

Thus, points of intersection are $(-2,0)$ and $(3,0)$.

Differentiating, $\frac{dy}{dx} = f'(x) = 2x - 1$

Slope of tangent at $(-2,0)$: $f'(-2) = 2(-2) - 1 = -5$.

Equation of tangent line at $(-2,0)$: $y - 0 = -5(x+2)$

$$5x + y + 10 = 0$$

Slope of tangent line at $(3,0)$: $f'(3) = 2(3) - 1 = 5$.

Similarly, equation of tangent at $(3,0)$: $y - 0 = 5(x-3)$

$$5x - y - 15 = 0$$

Hence, the equations of the tangent lines to the curve $y = x^2 - x - 6$ at the points of intersection of the curve with the x-axis are $5x + y + 10 = 0$ and $5x - y - 15 = 0$.

2.6. Derivatives of Higher Order

If $y = f(x)$ is a differentiable function of x, then, y' or f' is sometimes termed the first derivative of y with respect to x. If y' is a differentiable function, then, $\frac{d}{dx}y' = y'' = f''$ (read as "y double prime"). This is called the second derivative of y with respect to x. Likewise, $\frac{d}{dx}(y'') = y''' = f'''$ is the third derivative of y with respect to x, provided y'' exists. Moreover, $\frac{d}{dx}y^{n-1} = y^{(n)} = f^{(n)}$ is the nth derivative of y with respect to x, where n is a positive integer greater than 1.

Based on Leibniz notation, $\frac{dy}{dx}$ is the first derivative, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. The nth derivative of y with respect to x has the notation $\frac{d^ny}{dx^n}$ or $D_x{}^n y$ or $\frac{d^n f(x)}{dx^n}$.

Example 11. Given: $y = 2x^5 - 3x^4 - 6x^3 - x^2 - 1$. Find $y^{(4)}$.

$$y' = 10x^4 - 12x^3 - 18x^2 - 2x$$
$$y'' = 40x^3 - 36x^2 - 36x - 2$$
$$y''' = 120x^2 - 72x - 36$$
$$y^{(4)} = 240x - 72 = 24(10x - 3)$$

Example 12. Find $\frac{d^2y}{dx^2}$, given $y = \sqrt{x^2 + 4}$

Differentiate using the formula $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+4}}\frac{d}{dx}(x^2+4) = \frac{1}{2\sqrt{x^2+4}}(2x) = \frac{x}{\sqrt{x^2+4}}$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+4}}\right)$$

Using the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

$$\frac{d^2y}{dx^2}=\frac{\sqrt{x^2+4}(1)-x\left(\frac{1}{2\sqrt{x^2+4}}\right)(2x)}{\left(\sqrt{x^2+4}\right)^2}=\frac{\sqrt{x^2+4}-\frac{x^2}{\sqrt{x^2+4}}}{x^2+4}$$

$$\frac{d^2y}{dx^2}=\frac{\left(\sqrt{x^2+4}\right)^2-x^2}{\sqrt{x^2+4}}\cdot\frac{1}{\left(x^2+4\right)}=\frac{x^2+4-x^2}{\left(x^2+4\right)\sqrt{x^2+4}}=\frac{4}{\sqrt{\left(x^2+4\right)^3}}$$

But observe that from the given, the $y=\sqrt{x^2+4}$. Substitute on the above derivative, therefore, $\frac{d^2y}{dx^2}=\frac{4}{y^3}$

2.7. Implicit Differentiation

If the given function takes the form $F(x,y)=0$, we find $\frac{dy}{dx}$ by following the steps listed below.

a). Whenever possible, we solve the given equation of the curve for y and then, differentiate y with respect to x. This is true only for very simple equations; for complicated functions, this step is to be avoided.

b). Considering y as a function of x, differentiate each term of the given equation with respect to x and solve the resulting equation for y'.

Example 13. Find y', given: $y^2-3x^3y-5x^2+4=0$.

Implicitly differentiating, $2y\frac{dy}{dx}-3\left[x^3\frac{dy}{dx}+y\left(3x^2\right)\right]-10x=0$

$$2y\frac{dy}{dx}-3x^3\frac{dy}{dx}-9x^2y-10x=0$$

$$\frac{dy}{dx}\left(2y-3x^3\right)=10x+9x^2y$$

$$\frac{dy}{dx}=\frac{10x+9x^2y}{2y-3x^3}=\frac{x(10+9xy)}{2y-3x^3}$$

Example 14. Find y'', given $2x^2-y^2=5$.

Use implicit differentiation: $\frac{d}{dx}\left(2x^2\right)-\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(5)$

$$4x-2y\frac{dy}{dx}=0$$

$$-2y\frac{dy}{dx}=-4x$$

$$\frac{dy}{dx}=\frac{4x}{2y}=\frac{2x}{y}, \text{ provided } (y \neq 0)$$

Differentiate $\frac{dy}{dx}$ or y' with respect to x to get $\frac{d^2y}{dx^2}=y''$.

$$\frac{d}{dx}(y')=\frac{d^2y}{dx^2}=y''=2\left[\frac{y\frac{dx}{dx}-x\frac{dy}{dx}}{y^2}\right]$$

However, recall that $\frac{dy}{dx}=\frac{2x}{y}$. Substitute this on the above y" equation.

$$y''=2\left[\frac{y-x\left(\frac{2x}{y}\right)}{y^2}\right]=\frac{2}{y^2}\left[\frac{y^2-2x^2}{y}\right]=\frac{2}{y^3}\left[y^2-2x^2\right]$$

But $y^2-2x^2=-(2x^2-y^2)$ and from the given, $2x^2-y^2=5$. Substitution of these on the above y" equation yields

$$y''=\frac{2}{y^3}\left[-(2x^2-y^2)\right]=-\frac{2}{y^3}(5)=-\frac{10}{y^3}, \qquad \text{Therefore, } y''=-\frac{10}{y^3}.$$

Example 15. Find slope and equation of the tangent line to curve $2x^2-3xy+2y^2=2$ at point $\left(-1,-\frac{3}{2}\right)$.

Implicitly differentiate to find $\frac{dy}{dx}$ which will give the slope of the tangent line at the point of tangency $\left(-1,-\frac{3}{2}\right)$.

$$\frac{d(2x^2)}{dx}-3\frac{d}{dx}(xy)+\frac{d(2y^2)}{dx}=\frac{d(2)}{dx}$$

$$4x-3x\frac{dy}{dx}-3y+4y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(4y-3x)=3y-4x$$

$$\frac{dy}{dx}=\frac{3y-4x}{4y-3x}$$

Hence, the slope of the tangent line at $\left(-1,-\frac{3}{2}\right)$ is equal to

$$m=\frac{dy}{dx}=\frac{3\left(-\frac{3}{2}\right)-4(-1)}{4\left(-\frac{3}{2}\right)-3(-1)}=\frac{-\frac{9}{2}+4}{-6+3}=\frac{\frac{-9+8}{2}}{-3}=\frac{-\frac{1}{2}}{-3}=\frac{1}{2}\left(\frac{1}{3}\right)=\frac{1}{6}.$$

Equation of the tangent line is obtained using the point-slope form $y-y_1=m(x-x_1)$:

$$y-\left(-\frac{3}{2}\right)=\frac{1}{6}[x-(-1)]$$

$$6y+9=x+1$$

$$x-6y-8=0$$

Hence, the slope and equation of the tangent line to curve $2x^2-3xy+2y^2=2$ at point $\left(-1,-\frac{3}{2}\right)$ are respectively $\frac{1}{6}$ and $x-6y-8=0$.

Example 16. At what point of the curve $xy=6$ is the slope of the tangent line equal to $-\frac{1}{6}$?

By implicit differentiation: $x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}=\frac{-y}{x}$

From the given condition, $\frac{dy}{dx}=-\frac{1}{6}$. Substitution of this on the above equation results to

$$-\frac{1}{6}=-\frac{y}{x} \qquad y=\frac{x}{6}$$

From the given equation of the curve, $y=\frac{6}{x}$, substitution on the above equation gives

$$\frac{6}{x}=\frac{x}{6}$$

$$x^2=36$$

$$x^2-36=0$$

$$(x-6)(x+6)=0$$

$$x-6=0 \text{ or } x+6=0$$

$$x=6 \qquad \text{or} \qquad x=-6$$

$$y=1 \qquad y=-1$$

Hence, at points $(6,1)$ and $(-6,-1)$, the slope of the tangent line is equal to $-\frac{1}{6}$.

2.8. Chain Rule of Differentiation

We are used to have $y = f(x)$ in finding the derivative of y with respect to x. If this is not the given case, the Chain Rule is one of the most important tools in differentiation. To find $\frac{dy}{dx}$, if:

1). $y = f(u)$ and $x = g(u)$	:	$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$
2). $y = f(u)$ and $u = g(x)$	:	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
3). $x = f(y)$	:	$\frac{dy}{dx} = -\frac{1}{\frac{dx}{dy}}$

Note: In Case (b), in many instances, it is conveniently possible to express $\frac{dy}{dx}$ in terms of x alone.

Example 17. Find $\frac{dy}{dx}$, given $y = t^3 - 3$, $t = \sqrt{2x-1}$

Solution:

This example falls under Case 2. Therefore, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$. Differentiate $y = t^3 - 3$ with respect to t, Thus, $\frac{dy}{dt} = 3t^2$. Now, differentiate $t = \sqrt{2x-1}$ with respect to x. Hence,

$$\frac{dt}{dx} = \frac{1}{2\sqrt{2x-1}}(2)$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2x-1}}$$

Therefore, $\frac{dy}{dx} = 3t^2\left(\frac{1}{\sqrt{2x-1}}\right)$.

However, from the given, $t = \sqrt{2x-1}$. Substitute on the above equation to simplify $\frac{dy}{dx}$ and have it expressed in terms of x alone.

Therefore, $\frac{dy}{dx} = 3\left(\sqrt{2x-1}\right)^2\left(\frac{1}{\sqrt{2x-1}}\right) = 3(2x-1)\left(\frac{1}{\sqrt{2x-1}}\right) = 3\sqrt{2x-1}$

Example 18. If $y=\left(w^3+1\right)^5$ and $x=\dfrac{2}{w+1}$, find $\dfrac{dy}{dx}$.

Solution:

This example falls under Case 1. Differentiate $y=\left(w^3+1\right)^5$ with respect to w.

$$\frac{dy}{dw}=5\left(w^3+1\right)^4\left(3w^2\right)=15w^2\left(w^3+1\right)^4$$

Using the differentiation formula $\dfrac{d}{dx}\left(\dfrac{c}{u}\right)=\dfrac{-c}{u^2}\dfrac{d}{dx}(u)$, differentiate x with respect to w. Hence, $\dfrac{dx}{dw}=\dfrac{-2}{(w+1)^2}(1)=\dfrac{-2}{(w+1)^2}$.

Therefore, $$\frac{dy}{dx}=\frac{\frac{dy}{dw}}{\frac{dx}{dw}}=\frac{15w^2\left(w^3+1\right)^4}{\frac{-2}{(w+1)^2}}=15w^2\left(w^3+1\right)^4\cdot\frac{(w+1)^2}{-2}$$

$$\frac{dy}{dx}=-\frac{15}{2}w^2(w+1)^2\left(w^3+1\right)^4$$

Example 19. Find equation of the tangent line to curve $x=\dfrac{y}{y^2-2}$ at point $(1,2)$.

Solution: Slope of the tangent line is given by $\dfrac{dy}{dx}$ which is equivalent to $\dfrac{1}{\frac{dx}{dy}}$.

This falls under Case 3. First, find $\dfrac{dx}{dy}$ and later take its reciprocal to have $\dfrac{dy}{dx}$.

$$\frac{dx}{dy}=\frac{(y^2-2)(1)-y(2y)}{(y^2-2)^2}$$
$$\frac{dx}{dy}=\frac{y^2-2-2y^2}{(y^2-2)^2}=\frac{-y^2-2}{(y^2-2)^2}$$

Hence, $\dfrac{dy}{dx}=\dfrac{\left(y^2-2\right)^2}{-y^2-2}=-\dfrac{\left(y^2-2\right)^2}{y^2+2}$

Hence, slope of the tangent line at point $(1,2)$ is

$$m=-\frac{\left[(2)^2-2\right]^2}{(2)^2+2}=\frac{(2)^2}{6}=-\frac{4}{6}=-\frac{2}{3}$$

Thus, equation of tangent line using $y-y_1=m(x-x_1)$ is

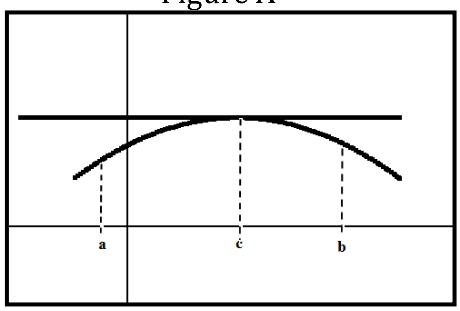
$$y-2=-\frac{2}{3}(x-1)$$
$$3y-6=-2x+2$$
$$2x+3y-8=0$$

2.9. Maximum and Minimum Value of a Function

Sketching the graph of function is better facilitated using the geometrical interpretation of the derivative of a function as the slope of the tangent line at a point to the graph of the function. The derivative serves as a great tool in determining at what point on the curve is the tangent line horizontal; that is, where the slope of the tangent line or y' equals zero.

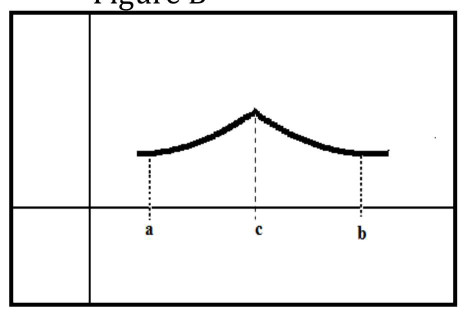
Definition: The function f is said to have a relative maximum value at c if there exists an open interval that contains c, on which f is defined such that $f(c) \geq f(x)$ for all x in this interval. Figures A and B each exhibit a sketch of a part of the graph of the function that has a relative maximum value at c.

Figure A



Relative Maximum: $f'(c) = 0$

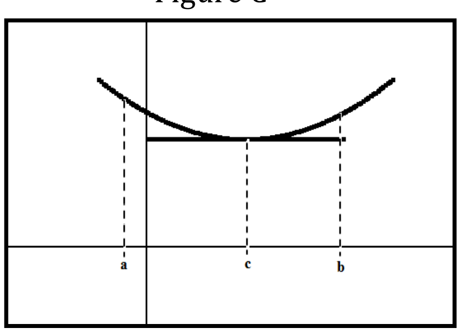
Figure B



Relative Maximum: $f'(c)$ is infinite (a cusp)

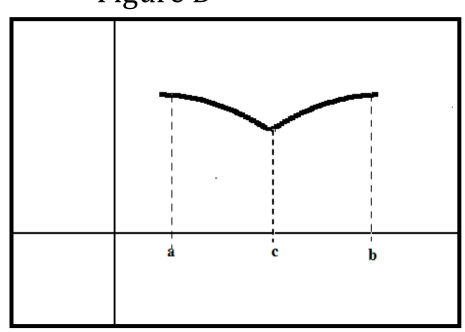
Definition: The function f is said to have a relative minimum value at c if there exists an open interval that contains c, on which f is defined such that $f(c) \leq f(x)$ for all x in this interval. Figures C and D shown below each exhibit a sketch of a part of the graph of the function that has a relative minimum value at c.

Figure C



Relative Minimum: $f'(c) = 0$

Figure D



Relative Minimum: $f'(c)$ is infinite (a cusp)

Definition: Relative maximum or minimum value of f occurs at critical points of f . A critical number of a function f is a value of x, say x_0, where $f'(x_0)=0$ or $f'(x_0)$ is undefined. A critical point of a function f is the point $(x, f(x))$ on the graph that corresponds to the critical number x. In Figures A and C on the preceding page, $f'(c)=0$ since tangent line at point where $x=c$ is horizontal, while in Figures B and D, $f'(c)$ is infinite since tangent line is vertical. In the succeeding discussions, relative maximum (minimum) value will simply be termed maximum (minimum) value.

2.9.1. Concavity Test

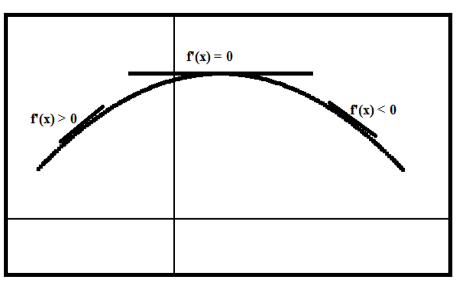
There are two possible tests that can be performed to determine and classify whether a critical point is a maximum, minimum point or neither maximum nor minimum.

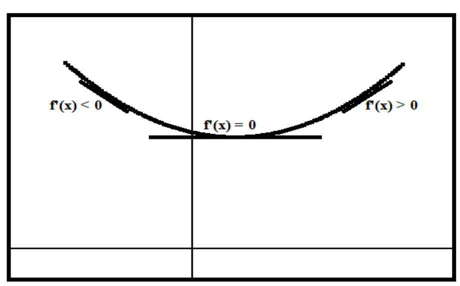
After finding the critical values by getting $f'(x)$ and equating it to zero, do the following steps to classify a critical value as to whether it is a maximum, minimum point or neither maximum nor minimum.

a). First-Derivative Test

1. Locate the critical numbers on the rectangular coordinate system to establish a number of intervals.
2. Determine the sign of $f'(x)$ on each interval.
3. If $x=x_o$ is a critical number, assume increasing value of x through $x=x_o$.

   Now, if $f'(x)$:
   - Changes from + to -, then, $x=x_o$ is a relative maximum value and concavity of the curve is downward.
   - Changes from - to +, then, $x=x_o$ is a relative minimum value and concavity of the curve is upward.
   - Does not change in sign, then, $x=x_o$ is neither a relative maximum nor a minimum value.





b). Second-Derivative Test

A critical number $x = x_0$ is

- a maximum value if $f''(x_0) < 0$.
- a minimum value if $f''(x_0) > 0$.

Note: If $f''(x_0) = 0$ or becomes infinite, the second-derivative test fails. In this case, the first-derivative test must be used.

Example 20. Determine and classify using either the first-derivative or the second derivative test the critical point/s of the curve represented by the given functions.

a). $y = 3x^5 - 20x^3$

b). $y = x^4 - 4x^3 - 2x^2 + 12x - 8$

c). $y = \dfrac{2x}{x^2 + 1}$

Solution:

a). $y = 3x^5 - 20x^3$

1. First, find the critical numbers of $f(x) = 3x^5 - 20x^3$ by finding $f'(x)$ using the power formula. It is best to express $f'(x)$ in the factored form. Here, we use common monomial factoring method.

$$f'(x) = 15x^4 - 60x^2$$
$$f'(x) = 15x^2(x^2 - 4)$$
$$f'(x) = 15x^2(x + 2)(x - 2)$$

2. Set the derivative equal to zero and solve for x.

$$15x^2(x + 2)(x - 2) = 0$$
$$15x^2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2$$

Hence, these values $x = 0, 2, -2$ are the critical numbers of f where the curve has horizontal tangents because at these numbers $f'(x)$ that is equal to the slope of the tangent has a value of zero.

After having determined the critical numbers, we need to classify whether the value is maximum, minimum or neither using either the first derivative or the second derivative test. To facilitate doing this task, draw a number line and put down the critical numbers found.

3. Let us use for this example the first derivative test to classify critical number $x=-2$. First assume a value of x slightly less than -2, say -3, and determine the sign of $f'(-3)$. Then, this time, assume another value of x more than -2, say -1, and note again the sign of $f'(-1)$.

$$f'(-3)=+(-)(-)=+$$
$$f'(-1)=+(+)(-)=-$$

Since $f'(x)$ changes sign from $+$ to $-$ as x increases through $x=-2$, then, $x=-2$ is a relative maximum or simply maximum. We do the same process at critical numbers $x=0$ and $x=2$.

At $x=0$: $f'(-1)=+(+)(-)=-$
$f'(1)=+(+)(-)=-$

Since $f'(x)$ changes sign from $-$ to $-$ as x increases through $x=0$, then, $x=0$ is neither maximum nor minimum.

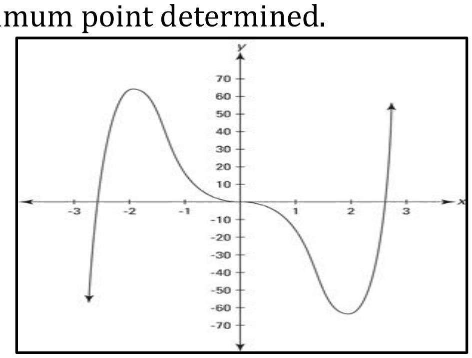
At $x=2$: $f'(1)=+(+)(-)=-$
$f'(3)=+(+)(+)=+$

Since $f'(x)$ changes sign from $-$ to $+$ as x increases through $x=2$, then, $x=2$ is a minimum.

4. Obtain the corresponding function values (the y-values) of the two relative extrema by substituting each critical number on the original function.

$$f(-2)=3(-2)^5-20(-2)^3=64$$
$$f(2)=3(2)^5-20(2)^3=64$$

Thus, the relative maximum is located at point $(-2,64)$ and the relative minimum is at point $(2,64)$. The graph of the given curve shown below reveals the maximum and minimum point determined.



b). $y=x^4-4x^3-2x^2+12x-8$

Solution: Find $f'(x)$ and equate it to zero.

$$f'(x) = 4x^3 - 12x^2 - 4x + 12 = 0$$

The right side of the above equation is factorable by grouping.

$$f'(x) = 4(x^3 - 3x^2 - x + 3) = 0$$
$$f'x) = x^3 - 3x^2 - x + 3 = 0$$
$$f'(x) = x^2(x-3) - (x-3) = 0$$
$$f'(x) = (x-3)(x^2-1) = 0$$
$$(x-3)(x+1)(x - y'(x) = 1) = 0$$

$x = 3 \qquad x = -1 \qquad x = 1$

$y = -17 \qquad y = -17 \qquad y = -1$

To classify the critical points $(3,-17), (-1,-17)$, and $(1,-1)$, we may perform either of the following tests:

a). First-Derivative Test:

$(3,-17)$: $\quad x < 3$: $x = 2.9$, $f'(2.9) = +(-)(+)(+) = -$

$\qquad x > 3$: $x = 3.1$, $f'(3.1)) = +(+)(+)(+) = +$

Since $f'(x)$ changes from $-$ to $+$ as x increases through $x = 3$, therefore, $(3,-17)$ is a minimum point.

$(-1,-17)$: $\quad x < -1$: $x = -1.1$, $f'(-1.1) = +(-)(-)(-) = -$

$\qquad x > -1$: $x = -0.9$, $f'(-0.9) = +(-)(+)(-) = +$

Since $f'(x)$ changes from $-$ to $+$ as x increases through $x = -1$, thus, $(-1,-17)$ is a minimum point.

$(1,-1)$: $\quad x < 1$: $x = 0.9$, $y'(0.9) = +(-)(+)(-) = +$

$\qquad x > 1$: $x = 1.1$, $y'(1.1) = +(-)(+)(+) = -$

With $f'(x)$ changing from $+$ to $-$ as x increases through $x = 1$, hence, $(1, -1)$is a maximum point of the curve.

b). Second-Derivative Test

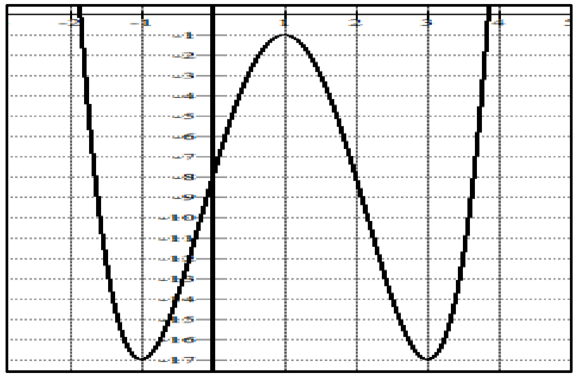
$$f''(x) = 12x^2 - 24x - 4$$
$$f''(x) = 4(3x^2 - 6x - 1)$$

$(3,-17)$: $f''(3) = +(+) = +$, $(3,-17)$ is a minimum point, concavity is upward

$(-1,-17)$: $f''(-1) = +(+) = +$, $(-1,-17)$ is a minimum point, concavity is downward

$(1,-1)$: $f''(1) = +(-) = -$, $(1,-1)$ is a maximum point, concavity is upward.

Observe that both tests done yield the same results. The graph on the next page of the given function reflects the maximum and minimum points of the given curve.



c). $y = \dfrac{2x}{x^2+1}$

Solution: Find the critical numbers by setting $f'(x)=0$.

$$f'(x) = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2}$$

$$0 = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2}$$

$$2(x^2+1) - 4x^2 = 0$$

$$2x^2 + 2 - 4x^2 = 0$$

$$2 - 2x^2 = 0$$

$$1 - x^2 = 0$$

$$(1+x)(1-x) = 0$$

$$x = 1 \qquad x = -1$$

$$y = 1 \qquad y = -1$$

Use the Second-derivative test to classify the critical points $(1,1)$ and $(-1,-1)$.

$$f''(x) = \frac{(x^2+1)^2(-4x) - (2-2x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-4x(x^2+1)[(x^2+1) + (2-2x^2)]}{(x^2+1)^2}$$

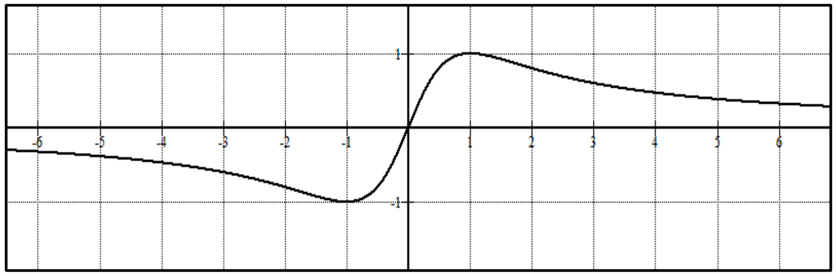
$$f''(x) = \frac{-4x(x^2+1)(3-x^2)}{(x^2+1)^4}$$

$$f''(x) = \frac{-4x(3-x^2)}{(x^2+1)^3}$$

$(1,1)$: $\quad f''(1) = \dfrac{-(+)(+)}{+} = -$, hence, $(1,1)$ is a maximum point.

$(-1,-1)$: $f''(x)=\dfrac{-(-)(+)}{+}=+$, therefore, $(-1,-1)$ is a minimum point.

The critical points are seen on the graph of the function shown below.



2.10.Optimization Problems

Words like biggest, largest, most, smallest, least, best, and others can be translated into mathematical language in terms of maxima and minima. One of the most important applications of derivative is illustrated on maximum/minimum optimization problems. Many students find this application intimidating because they are "word" problems, and no fixed pattern of solution exists to these problems. However, with their patience, they can minimize their anxiety and maximize their success with these problems by following the guidelines listed below:

1. Read the problem slowly and carefully. It is imperative to know exactly what the problem is asking. If appropriate, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out ones thought.

2. Identify the constant quantity in the given problem. Define variables to be used and carefully label the picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

3. Identify the quantity to be maximized or minimized and if it shall consist of more than one variable, express it in terms of one variable (if possible and practical) using the given conditions in the problem. Experience shows that most optimization problems begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. *Some problems may have no constraint equation. Some problems may have two or more constraint equations.*

4. Then differentiate using the well-known rules of differentiation.

5. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

Example 21. What positive number added to its reciprocal gives the minimum sum?

Let: S be the minimum sum

x = the required positive number

$\frac{1}{x}$ = the reciprocal of the number

Optimization equation: $S = x + \frac{1}{x} = x - x^{-1} = f(x)$

Note that this optimization problem has no constraint equation.

$$\frac{dS}{dx} = 1 - \frac{1}{x^2}$$

For S to be a minimum, $\frac{dS}{dx} = 0$.

$$0 = 1 - \frac{1}{x^2}$$

$$0 = \frac{x^2 - 1}{x^2}$$

$$0 = (x+1)(x-1)$$

$$x = 1 \quad \text{or} \quad x = -1\,\text{(Reject)}$$

Verification of the critical value using the second-derivative test.

$$S''(x) = 1 + x^{-2} = 1 + \frac{1}{x^2}$$

$$S''(1) = 1 + 1 = 2$$

Since S'' is positive or greater than zero when $x = 1$, then, the sum S is minimum at $x = 1$.

Example 22. Find two positive integers having a sum of 132 and the sum of their cubes has the minimum value.

Let: S be the minimum sum of their cubes

x be one positive number and y be the other number

Optimization equation: $S = x^3 + y^3$ ---------------- Equation (1)

Constraint equation: $x + y = 132$

$y = 132 - x$ ------------------ Equation (2)

Substitute Equation (2) into Equation (1) and differentiate with respect to x the result.

$$S = x^3 + (132 - x)^3 = f(x)$$
$$\frac{dS}{dx} = 3x^2 + 3(132 - x)^2(-1)$$
$$0 = 3\left[x^2 - (132 - x)^2\right]$$
$$0 = x^2 - \left(17424 - 264x + x^2\right)$$
$$0 = -17424 + 264x$$
$$264x = 17424$$
$$x = 66$$

Therefore, $y = 132 - x = 132 - 66 = 66$

Verification of the critical value using the second-derivative test:

$$y'(x) = 3x^2 - 3(132 - x)^2$$
$$y''(x) = 6x + 6(132 - x)$$
$$y''(66) = 6(66) + 6(132 - 66) = 792$$

Hence, the sum S is minimum at $x = 66$ since $y''(66)$ is greater than zero.

Example 23. If the product of the square of one number by the cube of the other is to be *the greatest*, find the two numbers if there sum equals 20.

Let: x be one number
z be the product of the square of one number by the cube of the other number

Optimization equation: $z = x^2 y^3$ ----------------- Equation (1)

Constraint equation: $x + y = 20$

$y = 20 - x$ ---------------- Equation (2)

Substitute Equation (2) into Equation (1).

$$z = x^2(20 - x)^3 = f(x)$$

Now, differentiate z with respect to x.

$$\frac{dz}{dx} = x^2 \frac{d}{dx}(20 - x)^3 + (20 - x)^3 \frac{d}{dx}\left(x^2\right)$$
$$\frac{dz}{dx} = x^2(3)(20 - x)^2(-1) + (20 - x)^3(2x)$$
$$\frac{dz}{dx} = x(20 - x)^2[-3x + 2(20 - x)]$$

For Z to be the greatest,

$$x = 0 \quad or \quad 20 - x = 0 \quad or \quad 40 - 5x = 0$$

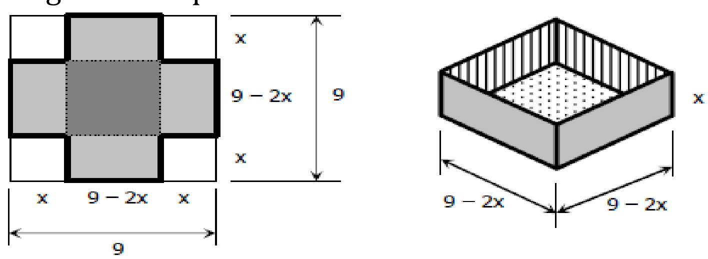
$$x = 0 \quad or \quad x = 20 \quad or \quad x = 8$$

Take note x cannot be both zero and 20; otherwise, the product z will have zero value. Hence, values $x = 0$ and $x = 20$ are rejected.

Therefore, the numbers are 8 and 12.

Example 24. A box is to be made of a piece of cardboard 9 inches square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Let: V be the volume of the largest box that can be made as described above
x be the edge of the square cut from each corner



Optimization equation: $V = LWH = x(9 - 2x)(9 - 2x) = x(9 - 2x)^2 = f(x)$
Take note that this optimization problem has no constraint equation.
Differentiate V with respect to x.

$$\frac{dV}{dx} = x(2)(9 - 2x)(-2) + (9 - 2x)^2(1)$$

$$\frac{dV}{dx} = -4x(9 - 2x) + (9 - 2x)^2$$

$$0 = (9 - 2x)(-4x + 9 - 2x)$$

$$0 = (9 - 2x)(9 - 6x)$$

Equate each factor to zero and solve for the value of x.

$$9 - 2x = 0 \quad \text{or} \quad 9 - 6x = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{9}{6} = \frac{3}{2}$$

The value $x = \frac{9}{2}$ is rejected since the edge (9 – 2x) of the box becomes zero at this value of x. Therefore, the edge of the square cut from each corner is $x = \frac{3}{2}$ inches. Substitution of this x value on the volume equation results to

$$V=\frac{3}{2}\left[9-2\left(\frac{3}{2}\right)\right]^2=\frac{3}{2}(9-6)^2=\frac{3}{2}(36)=54$$

Therefore, the volume of the largest box that can be made as described above is 54 inches^3

Example 25. A piece of wire 40 cm long is to be cut into two pieces. One piece will be bent to form a circle; the other will be bent to form a square. Find the lengths of the two pieces that cause the sum of the area of the circle and square to be a minimum.

Let: x be the part of the wire bent to form a square of edge s

$(40-x)$ be the other part of the wire bent into a circle of radius r

For the square part of the wire, $P=4s$

$$x=4s$$

$$s=\frac{x}{4}$$

For the circular part, $C=2\pi r$

$$40-x=2\pi r$$

$$r=\frac{40-x}{2\pi}$$

The sum of their areas: $A=s^2+\pi r^2$

$$A=\left(\frac{x}{4}\right)^2+\pi\left(\frac{40-x}{2\pi}\right)^2$$

$$A=\frac{x^2}{16}+\frac{1}{4\pi}(40-x)^2$$

Differentiate A with respect to x. $\frac{dA}{dx}=\frac{1}{16}(2x)+\frac{1}{4\pi}(2)(40-x)(-1)$

$$\frac{dA}{dx}=\frac{x}{8}-\frac{1}{2\pi}(40-x)$$

$$0=\frac{x}{8}-\frac{1}{2\pi}(40-x)$$

$$0=\frac{\pi x-4(40-x)}{8\pi}$$

$$0=\frac{\pi x-160+4x}{8\pi}$$

$$0=\frac{x(\pi+4)-160}{8\pi}$$

$$0 = x(\pi + 4) - 160$$

$$x = \frac{160}{4\pi} = 22.4 \text{ cm}$$

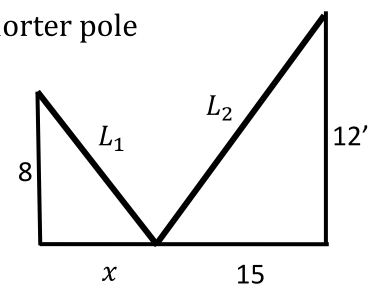
Therefore, $40 - x = 40 - 22.4 = 17.6$ cm

And, $40 - x = 40 - 22.4 = 17.6$ cm

Therefore, the lengths of the wire that will give the minimum combined area of the circle and the square are 22.4 cm and 17.6 cm.

Example 26. Two posts, one 8 feet high and the other 12 feet high, stand 15 feet apart. They are to be supported by wires attached to a single stake at ground level. Where the stake should be placed so that the least amount of wire is used?

Let: x be the distance of the stake from the shorter pole



From the given figure at the right, $L_1 = \sqrt{(8)^2 + x^2}$

Likewise, $L_2 = \sqrt{(12)^2 + (15 - x)^2}$

Thus, total length of wires used $= L = L_1 + L_2$

$$L = \sqrt{(8)^2 + x^2} + \sqrt{(12)^2 + (15 - x)^2}$$

Differentiate. $\frac{dL}{dx} = \frac{1}{2\sqrt{(8)^2 + x^2}}(2x) + \frac{1}{2\sqrt{(12)^2 + (15 - x)^2}}(2)(15 - x)(-1)$

$$0 = \frac{x}{\sqrt{64 + x^2}} - \frac{15 - x}{\sqrt{144 + (15 - x)^2}}$$

$$0 = \frac{x\sqrt{144 + (15 - x)^2} - (15 - x)\sqrt{64 + x^2}}{\sqrt{54 + x^2}\left(\sqrt{144 + (15 - x)^2}\right)}$$

$$0 = x\sqrt{144 - (15 - x)^2} - (15 - x)\sqrt{64 + x^2}$$

Solve the radical equation. $(15 - x)^2(64 + x^2) = x^2\left[144 + (15 - x)^2\right]$

$$(15 - x)^2(64 + x^2) = x^2\left[144 + (225 - 30x + x^2)\right]$$

$$64(225 - 30x + x^2) + x^2(225 - 30x + x^2) = x^2(369 - 30x + x^2)$$

$$1400 - 1920x + 64x^2 + 225x^2 - 30x^3 + x^4 = 369x^2 - 30x^3 + x^4$$

$$0 = 80x^2 + 1920x - 14400$$

$$0 = x^2 + 24x - 180$$

$$0 = (x - 6)(x + 30)$$

$$x = 6 \quad or \quad x = -30 \text{ Rejected}$$

Thus, the stake should be positioned 6 m from the shorter post or 9 m from the longer post.

2.11. Optimization Problem Using Implicit Differentiation

On the following optimization problems, observe how some of them are solved using two methods of solution and one of which is by implicit differentiation.

Example 27. Resolve Example 23 by using implicit differentiation.

Let: x and y be the numbers

Constraint Equation: $x + y = 20$ ------------------- Equation (1)

Optimization Equation: $z = x^2 y^3$ ------------------- Equation (2)

Here, there is no need to express Z in terms of one variable. Rather, to get $\frac{dz}{dx}$, we use implicit differentiation and equate the derivative to zero.

$$\frac{dZ}{dx} = x^2\left(3y^2\frac{dy}{dx}\right) + y^3(2x)$$

$$\frac{dZ}{dx} = 3x^2y^2\frac{dy}{dx} + 2xy^3$$

$$\frac{dZ}{dx} = xy^2\left(3x\frac{dy}{dx} + 2y\right)$$

$$0 = xy^2\left(3x\frac{dy}{dx} + 2y\right)$$

$$\frac{dy}{dx} = -\frac{2y}{3x} \text{ ------------------------- Equation (3)}$$

Likewise, differentiate Equation (1) implicitly with respect to x.

$$1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1 \text{ ------------------ Equation (4)}$$

Substitute Equation (4) into Equation (3).

$$-1 = \frac{-2y}{3x}$$

$$y = \frac{3x}{2} \text{ -------------------- Equation (5)}$$

Substitute Equation (5) into Equation (1) to get an equation in terms of one variable. Solve the resulting equation for the value of the variable.

$$x + \frac{3x}{2} = 20$$

$$2x + 3x = 40$$

$$5x = 40$$

$$x = 8$$

Substitution into Equation (1) or Equation (5) will give the corresponding value of y. Hence, $y = 20 - 8 = 12$. Therefore, the numbers are 8 and 12.

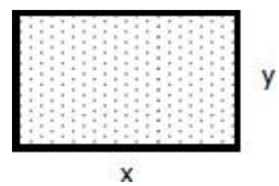
Example 28. A rectangular lot of area 150 m^2. What should be the dimensions of the lot for it to be enclosed by the least amount of fencing?

Let: x and y be the dimensions of the rectangular lot

P be the perimeter of the lot which is equal to the least amount of fencing needed to enclose the lot.

A be the area of the lot

$A = 150\ m^2$ (a constant)



Method (1). Area of rectangular is given by equation $A = xy$.

Constraint equation: $150 = xy$ ----------------- Equation (A)

$$y = \frac{150}{x}$$ ----------------- Equation (1)

Optimization equation: $P = 2x + 2y$ ------------- Equation (2)

Substitute Equation (1) into Equation (2):

$$P = 2x + 2\left(\frac{150}{x}\right)$$

$$P = 2x + \frac{300}{x} = 2x + 300\,x^{-1} = f(x)$$

Differentiate P with respect to variable x and equate its derivative to zero for P to be the least amount of needed fencing.

$$\frac{dP}{dx} = 2 - 300\,x^{-2}$$

$$0 = 2 - 300\,x^{-2}$$

$$x^2 = 150$$

$$x = \sqrt{150} = \sqrt{25(6)} = 5\sqrt{6}$$

Substitute into Equation (1): $y = \frac{150}{\sqrt{150}} = \sqrt{150} = 5\sqrt{6}$

Therefore, the rectangular lot is a square measuring $5\sqrt{6}$ m by $5\sqrt{6}$ m. Hence, the lot is square in shape.

Method (2). Resolve the problem using implicit differentiation.
Differentiate Equation (A) with respect to x.

$$0 = x\frac{dy}{dx} + y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$ -------------- Equation (3)

Differentiate Equation (2) with respect to x, substitute Equation (3) into the resulting equation and, then, equate the derivative of P to zero.

$$\frac{dP}{dx} = 2 + 2\frac{dy}{dx}$$ -------- Equation (4)

$$\frac{dP}{dx} = 2 + 2\left(-\frac{y}{x}\right)$$

$$0 = 2 - \frac{2y}{x}$$

$$y = x$$ ------------------- Equation (5)

Substitute Equation (5) into Equation (1).

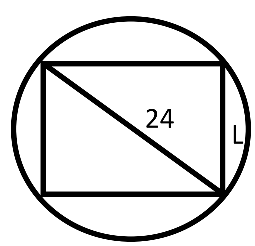
$$x = \frac{150}{x}$$

$$x^2 = 150$$

$$x = y = \sqrt{150} = 5\sqrt{6}$$

Example 29. The strength of a rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of diameter 24 inches.

Let: W be the breadth of the beam and L be its depth
S be the strength of the beam



Optimization equation: $S = WL^2$

Differentiate S implicitly with respect to W.

$$\frac{dS}{dW} = W\left(2L\frac{dL}{dW}\right) + L^2(1)$$

For S to be the greatest, $\frac{ds}{dw} = 0$.

$$0 = 2LW\frac{dL}{dW} + L^2$$ ------------------------Equation (1)

By Pythagorean Theorem, $(24)^2 = W^2 + L^2$ -----------------Equation (2)

Differentiate implicitly with respect to x.

$$0 = 2W + 2L\frac{dL}{dW}$$

$$\frac{dL}{dW} = -\frac{W}{L}$$ ---------------------- Equation (3)

Substitute Equation (3) into Equation (1).

$$0 = 2LW\left(-\frac{W}{L}\right) + L^2$$

$$0 = -2W^2 + L^2$$

$$L^2 = 2W^2$$

$$L = \sqrt{2}W$$

Therefore, for the beam to be the strongest one, the depth L must be $\sqrt{2}$ times the breadth W.

Example 30. Find the shortest distance from the point $(5,0)$ to the curve $2y^2 = x^3$.

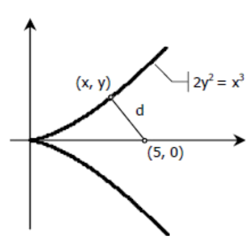
Let: (x, y) be the point on the curve $2y^2 = x^3$ nearest to point $(5,0)$

D the shortest distance from to point (x, y) to point $(5,0)$

Method (1). Use distance formula between two points and use implicit differentiation with respect to x.

Optimization equation: $D = \sqrt{(x-5)^2 + y^2}$ ------------ Equation (A)

$$\frac{dD}{dx} = \frac{1}{2\sqrt{(x-5)^2}}\left[2(x-5) + 2y\frac{dy}{dx}\right]$$



$$0 = \frac{1}{2\sqrt{(x-5)^2}}\left[2(x-5) + 2y\frac{dy}{dx}\right]$$

$$0 = 2(x-5) + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{(x-5)}{y}$$ -------------- Equation (1)

Constraint equation: $2y^2 = x^3$

Differentiate implicitly the given equation of the curve with respect to x.

$$4y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{4y} \text{ -------------- Equation (2)}$$

Substitute Equation (1) into Equation (2).

$$-\frac{(x-5)}{y} = \frac{3x^2}{4y}$$

$$-4(x-5) = 3x^2$$

$$3x^2 = -4x + 20$$

$$3x^2 + 4x - 20 = 0$$

$$(3x+10)(x-2) = 0$$

$$3x + 10 = 0 \quad or \quad x - 2 = 0$$

$$x = -\frac{10}{3} \quad or \quad x = 2$$

We reject $x = -\frac{10}{3}$ since corresponding y-value is imaginary. Thus, we use $x = 2$ and substitute it on the given equation of the curve to get the corresponding y-value. Therefore,

$$2y^2 = (2)^3$$

$$2y^2 = 8$$

$$y^2 = 4$$

$$y = \pm 2$$

Thus, the shortest distance $D = \sqrt{(2-5)^2 + 4} = \sqrt{9+4} = \sqrt{13}$

Method (2). From the given equation of the curve, $y^2 = \frac{x^3}{2}$. Substitute this on Equation (A), differentiate D with respect to x and set $\frac{dD}{dx} = 0$.

$$D = \sqrt{(x-5)^2 + \frac{x^3}{2}}$$

$$D = \sqrt{2(x-5)^2 + x^3}$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{2(x-5)^2 + x^3}}\left[2(2)(x-5)(1) + 3x^2\right]$$

$$0 = 4(x-5) + 3x^2$$

$$3x^2 + 4x - 20 = 0$$

$$(3x+10)(x-2) = 0$$

$$3x + 10 = 0 \quad or \quad x - 2 = 0$$

$$x = -\frac{10}{3} \quad or \quad x = 2$$

Compare the results of Method (1) with those of Method (2). Thus, the shortest distance $D = \sqrt{13}$.

2.12. Time – Rates

If the value of a variable y depends on the time t, then, $\frac{dy}{dt}$ is called its time-rate or rate of change with respect to time. When two or more quantities, all functions of t, are related by an equation, the relation between their time-rates may be attained by differentiating both sides of the equation with respect to time t. Basic time-rates are velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$. If the time-rate $\frac{dy}{dt}$ is positive, it means the quantity y is increasing with time.

Steps in Solving Time Rates Problem

1. Identify what quantities are changing and what are fixed with time.
2. Assign variables to those that are changing and appropriate value (constant) to those that are fixed.
3. Find an equation relating all the variables and constants in Step 2.
4. Differentiate the equation with respect to time.

Example 31. Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Find the rate at which the radius of the balloon is increasing when the radius of the balloon is 10 cm.

Solution: Both the volume V and the radius r of the balloon will change with time while the balloon is being inflated. That is, they are both functions of time. Hence, we have $V(t)$ and $r(t)$. We need to relate these two quantities to each other and that is with the formula for the volume of a sphere. It is given that $\frac{dV}{dt} = +5 \text{ cm}^3/\text{min}$ and we wish to know the time-rate of r represented by $\frac{dr}{dt}$ when $r = 10$ cm. Observe that $\frac{dV}{dt}$ is given a positive sign because volume increases while the balloon is being inflated.

Volume of sphere is given by equation $V = \frac{4}{3}\pi r^3$ ----------- Equation (1)

Recall that the rate of change of the radius is nothing more than the derivative of r with respect to t. We differentiate (1) with respect to t.

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

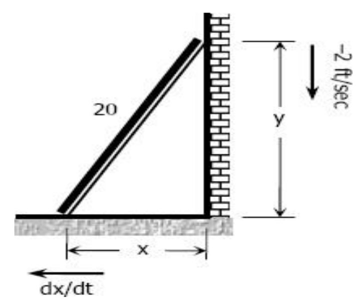
Substitute the given information.

$$+5 = 4\pi(10)^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{4(100)\pi} = \frac{1}{4(20)\pi} = \frac{1}{80\pi} \text{ cm/min}$$

Therefore, the time-rate of radius of the balloon is increasing at a rate of $\frac{1}{80\pi}$ cm/min.

Example 32. A ladder 20 feet long leans against a vertical wall. If the top slides downward at the rate of 2 feet/sec, find how fast the lower end is moving when it is 16 feet from the wall.



Solution: The length of the ladder remains constant with time at a value of 20 feet. Looking at the given figure, the quantities changing with time are the distance x of the foot or lower end of the ladder from the wall and the distance y of the top of the ladder from the ground. The given information is the time-rate of y, that is, $\frac{dy}{dt} = -2$ feet/sec which is given a negative sign since y decreases with time and we wish to know the time-rate of x, that is $\frac{dx}{dt}$ when $x = 16$ feet.

The equation that relates quantities changing with time is obtained using Pythagorean Theorem. $y^2 + x^2 = (20)^2$ -------------- Equation (1)

Differentiate with respect to time. $2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$

$y\frac{dy}{dt} + x\frac{dx}{dt} = 0$ ------------------ Equation (2)

Substitute $x = 16$ into Equation (1) get the corresponding value of y.

$$y^2 + (16)^2 = (20)^2$$

$$y^2 = 400 - 256 = 144$$

$$y = 12 \text{ feet}$$

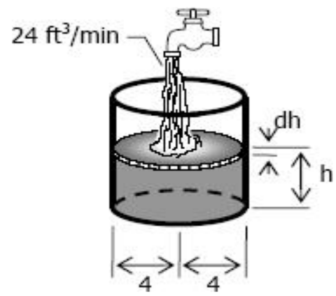
Substitute into Equation (2). $12(-2)+16\dfrac{dx}{dx}=0$

$$16\frac{dx}{dt}=24$$

$$\frac{dx}{dt}=\frac{24}{16}=\frac{3}{2}\text{ ft/sec}$$

Take note that the computed $\dfrac{dx}{dt}$ is positive since x is increasing with time.

Example 33.Water is flowing into a vertical cylindrical tank at the rate of 24 ft^3/min. If the radius of the tank is 4 feet, how fast is the surface rising?



Solution: The quantities that are changing with time are the depth h and the volume V of the water in the cylindrical tank. The radius of the tank remains constant with time at a value of 4 feet. The required quantity in this problem is the time-rate of h or $\dfrac{dh}{dt}$.

The time-rate of volume $\dfrac{dV}{dt}=+24$ ft^3/min; it is positive, just like time-rate of depth h, since the volume of water in the tank increases with time.

At any time t, volume of water in the cylindrical can is given by equation

$$V=\pi r^2 h=\pi(4)^2 h=16\pi h$$

Differentiate both sides of the equation above with respect to time.

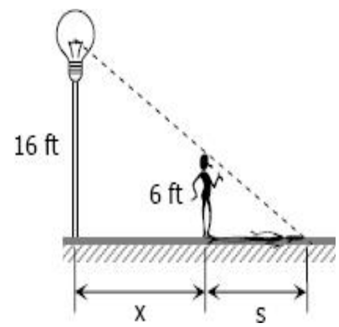
$$\frac{dV}{dt}=16\pi\frac{dh}{dt}$$

Substitute the given information. $+24=16\pi\dfrac{dh}{dt}$

$$\frac{dh}{dt}=\frac{24}{16\pi}=\frac{3}{2\pi}\text{ ft/min}$$

Example 34. A man 6 feet tall walks away from a lamp post 16 feet high at the rate of 5 miles per hour. How fast does his shadow lengthen?

Solution: Let the length of the man's shadow at any time be S and his distance from the lamp post be x; the given time-rate is $\dfrac{dx}{dt}=+5$ miles/hr; the required time-rate is $\dfrac{dS}{dt}$.



By similar triangle: $\frac{6}{s}=\frac{16}{s+x}$

$$6(s+x)=16s$$
$$6s+6x=16s$$
$$10s=6x$$
$$s=\frac{6}{10}x=\frac{3}{5}x$$

Differentiate s with respect to time t.

$$\frac{ds}{dt}=\frac{3}{5}\frac{dx}{dt}=\frac{3}{5}(+5)=3\frac{\text{miles}}{\text{hr}}$$

Hence, the shadow of the man is lengthening at the rate of 3 $\frac{\text{miles}}{\text{hr}}$.

2.13. Derivatives of Transcendental Functions

Transcendental function is a function that is not an algebraic function. Transcendental functions include trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions.

A. Derivatives of Trigonometric Functions

Let u be a differentiable function of x.

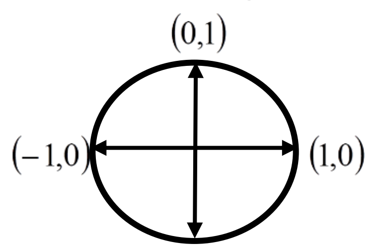
1. $\frac{d}{dx}\sin u=\cos u\frac{du}{dx}$	4. $\frac{d}{dx}\cot u=-\csc^2 u\frac{du}{dx}$
2. $\frac{d}{dx}\cos u=-\sin u\frac{du}{dx}$	5. $\frac{d}{dx}\sec u=\sec u\tan u\frac{du}{dx}$
3. $\frac{d}{dx}\tan u=\sec^2 u\frac{du}{dx}$	6. $\frac{d}{dx}\csc u=-csu\cot u\frac{du}{dx}$

It is but wise to review some basic concepts and commonly used trigonometric relations/identities that facilitate the differentiation process and the simplification of derivatives of trigonometric functions.

A.1. Trigonometric Functions of Angle A with Point $P(x,y)$ on Its Terminal Side

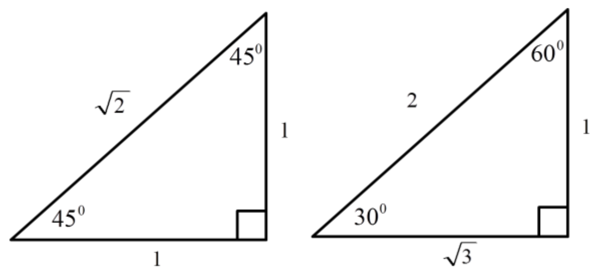
1. $\sin A=\frac{y}{r}$	4. $\csc A=\frac{r}{y}$
2. $\cos A=\frac{x}{r}$	5. $\sec A=\frac{r}{x}$
3. $\tan A=\frac{y}{x}$	6. $\cot A=\frac{x}{y}$

A.2. Trigonometric Functions of Quadrantal Angles



A.3. Trigonometric Functions of Special Angles

Use the SOH-CAH-TOA definitions of the trigonometric functions of the special acute angles.



A.4. Basic Relations for Trigonometric Functions

1. $\tan A = \dfrac{\sin A}{\cos A}$	5. $\cot A = \dfrac{1}{\tan A}$
2. $\cot A = \dfrac{\cos A}{\sin A}$	6. $\sin^2 A + \cos^2 A = 1$
3. $\sec A = \dfrac{1}{\cos A}$	7. $1 + \tan^2 A = \sec^2 A$
4. $\csc A = \dfrac{1}{\sin A}$	8. $1 + \cot^2 A = \csc^2 A$

A.5. Trigonometric Functions of Negative Angle

1. $\sin(-A) = -\sin A$	3. $\tan(-A) = -\tan A$	5. $\sec(-A) = \sec A$
2. $\cos(-A) = \cos A$	4. $\csc(-A) = -\csc A$	6. $\cot(-A) = -\cot A$

A.6. Sum and Difference of Two Angles Identities

1. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
2. $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$
3. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

A.7. Double Angle Identities

1. $\sin 2A = 2\sin A \cos A$	4. $\cos 2A = 1 - 2\sin^2 A$
2. $\cos 2A = \cos^2 A - \sin^2 A$	5. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$
3. $\cos 2A = 2\cos^2 A - 1$	6. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

Example 35. Find $\frac{dy}{dx}$ of the given functions.

a) $y = \sin\left(2x^2 - 1\right)$

Solution: Use $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$ where: $u = 2x^2 - 1$ and $\frac{du}{dx} = 4x$.

Hence, $\frac{dy}{dx} = \cos\left(2x^2 - 1\right)\cdot(4x) = 4x\cos\left(2x^2 - 1\right)$

b) $y = \cos\left(x^2 + 4\right)^3$

Solution: Use $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

where: $u = \left(x^2 + 4\right)^3$ and $\frac{du}{dx} = 3\left(x^2 + 4\right)^2\cdot(2x) = 6x\left(x^2 + 4\right)^2$.

Therefore, $\frac{dy}{dx} = -\sin\left(x^2 + 4\right)^3\left[6x\left(x^2 + 4\right)^2\right] = -6x\left(x^2 + 4\right)^2\sin\left(x^2 + 4\right)^3$.

c) $y = \csc\left(\frac{1}{x}\right)$

Solution: Use $\frac{d}{dx}\csc u = -csu \cot u \frac{du}{dx}$ where: $u = \frac{1}{x}$ and $\frac{du}{dx} = -\frac{1}{x^2}$.

Hence, $\frac{dy}{dx} = -\csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)$.

d) $y = \cot\sqrt{2x}$

Solution: Use $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$ where: $u = \sqrt{2x}$ and $\frac{du}{dx} = \frac{1}{2\sqrt{2x}}\cdot(2) = \frac{1}{\sqrt{2x}}$.

Therefore, $\frac{dy}{dx} = -\csc^2\sqrt{2x}\cdot\left(\frac{1}{\sqrt{2x}}\right) = -\frac{1}{\sqrt{2x}}\csc^2\sqrt{2x}$.

e) $y = \frac{1}{2}\tan x \sin 2x$

Solution: Use the product formula $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

where: $u = \tan x$ $\qquad v = \sin 2x$

$\frac{du}{dx} = \sec^2 x$ $\qquad \frac{dv}{dx} = (\cos 2x)(2) = 2\cos 2x$

Therefore, $\frac{dy}{dx} = \frac{1}{2}\left[(\tan x)(2\cos 2x) + (\sin 2x)(\sec^2 x)\right]$

$$\frac{dy}{dx} = \tan x \cos 2x + \frac{1}{2}\sin 2x \sec^2 x$$

However, the above derivative can further be simplified using the appropriate trigonometric identities. That is, $\tan x = \frac{\sin x}{\cos x}$, $\cos 2x = 2\cos^2 x - 1$, $\sin 2x = 2\sin x \cos x$ and $\sec x = \frac{1}{\cos x}$.

Hence, $\frac{dy}{dx} = \frac{\sin x}{\cos x}(2\cos^2 x - 1) + \frac{1}{2}(2\sin x\cos x)\left(\frac{1}{\cos x}\right)^2$

$$\frac{dy}{dx} = 2\left(\frac{\sin x}{\cos x}\right)\cos^2 x - \frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = 2\sin x \cos x$$

$$\frac{dy}{dx} = \sin 2x$$

f) $y = \frac{1 - \cos 2x}{1 + \cos 2x}$

Solution: Use the quotient formula $\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

where: $u = 1 - \cos 2x$ $\qquad v = 1 + \cos 2x$

$\frac{du}{dx} = -(-\sin 2x)(2) = 2\sin 2x$ $\qquad \frac{dv}{dx} = (-\sin 2x)(2) = -2\sin 2x$

Thus, $\frac{dy}{dx} = \frac{(1+\cos 2x)(2\sin 2x) - (1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$

$$\frac{dy}{dx} = \frac{2\sin 2x + 2\sin 2x\cos 2x + 2\sin 2x - 2\sin 2x \cos 2x}{(1+\cos 2x)^2}$$

$$\frac{dy}{dx} = \frac{4\sin 2x}{(1+\cos 2x)^2}$$

g) $y = \sec^4(x^3)$

Solution: Use the general power formula $\frac{d}{dx} cu^n = cnu^{n-1} \frac{du}{dx}$,

where: $n = 4$, $u = \sec x^3$ and $\frac{du}{dx} = (\sec x^3 \tan x^3)(3x^2)$

Therefore, $\frac{dy}{dx} = 4(\sec x^3)^3 (3x^2)(\sec x^3 \tan x^3)$

$$\frac{dy}{dx} = 12x^2 \sec^4 x^3 \tan x^3$$

But, $y = \sec^4 x^3$. Hence, $\frac{dy}{dx} = 12x^2 y \tan x^3$.

h) $y = \sin(x + y)$

Solution: Use Implicit differentiation.

$$y' = \cos(x + y)(1 + y')$$
$$y' = \cos(x + y) + y' \cos(x + y)$$
$$y' - y' \cos(x + y) = \cos(x + y)$$
$$y'[1 - \cos(x + y)] = \cos(x + y)$$
$$y' = \frac{\cos(x + y)}{1 - \cos(x + y)}$$

i) $x\cos y + y\cos x = 2$

Solution: Use implicit differentiation.

$$x\left[-\sin y \frac{dy}{dx}\right] + \cos y + y[-\sin x] + \cos x\left(\frac{dy}{dx}\right) = 0$$
$$-x\sin x \frac{dy}{dx} + \cos x \frac{dy}{dx} = y\sin x - \cos y$$
$$\frac{dy}{dx}(-x\sin x + \cos x) = y\sin x - \cos y$$
$$\frac{dy}{dx} = \frac{y\sin x - \cos y}{-x\sin x + \cos x}$$

Example 36. Find equation of the tangent line to the curve $y = x\cos x$ at the point where $x = \pi$.

Solution: To get the equation of the tangent line, we need to know its slope and the point of tangency.

Find the slope the tangent by differentiating the given function. Hence,

$$\frac{dy}{dx} = x(-\sin x) + \cos x = -x\sin x + \cos x$$

At $x=\pi$: $\frac{dy}{dx}=-\pi(\sin\pi)+\cos\pi=-\pi(0)+(-1)=-1$

Substitute $x=\pi$ to the given equation of the curve to get the corresponding value of y.

$$y=\pi\cos\pi=\pi(-1)=-\pi$$

Thus, point of tangency is at $(\pi,-\pi)$

Use the point-slope form to get equation of the tangent line at $(\pi,-\pi)$.

$$y-(-\pi)=-1(x-\pi)$$
$$y+\pi=-x+\pi$$
$$y=-x$$

Therefore, at $x=\pi$, $y=-x$ is the equation of the tangent line to curve $y=x\cos x$.

Example 37. At what point does the tangent line to the curve $y=4\tan 2x$ parallel to the line $y-8x+3=0$?

Solution: First, differentiate the given function and equate the resulting derivative to the slope of the given line. They are parallel lines, so their slopes are equal. Solve for the value of x and the corresponding y-value.

$$y'=4(\sec^2 2x)(2)=8\sec^2 2x$$

At the unknown point, $y'=m_{TL}=8$.

$$8=8\sec^2 2x$$
$$\sec^2 2x=1$$
$$(\sec 2x+1)(\sec 2x-1)=0$$
$$\sec 2x=1 \qquad \sec 2x=-1$$
$$2x=0,\quad 2\pi \qquad 2x=\pi,\quad 3\pi$$
$$x=0,\quad \pi \qquad x=\frac{\pi}{2},\frac{3\pi}{2}$$

Substitute the x-value on equation $y=4\tan 2x$.

When $x=0$, $\quad y=\tan 0=0$.

When $x=\pi$, $\quad y=\tan\pi=0$.

When $x=\frac{\pi}{2}$, $\quad y=\tan\pi=0$.

When $x = \frac{3\pi}{2}$, $y = \tan 3\pi = 0$.

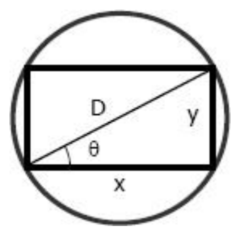
Therefore, the four points on the curve $y = 4\tan 2x$ where the tangent line is parallel to line $y - 8x + 3 = 0$ are $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 0)$, and $\left(\frac{3\pi}{2}, 0\right)$.

Optimization Problems Involving Trigonometric Functions

The use of trigonometric functions facilitates solution to many maxima-minima applications.To do it, identify the constant terms and the quantity/variable to be maximized (or minimized), differentiate that quantity/variable, equate the derivative to zero and then, solve for the value of the variable left on the resulting equation.

Example 38. Find the shape of the rectangle of maximum perimeter inscribed in a circle of diameter D.

Solution: Let x and y be the breadth and length of the rectangle whose perimeter P needs to be the maximum; θ be the acute angle the diameter makes with the breadth of the rectangle, as shown on the accompanying figure,. The diameter D of the circle is a constant quantity in this problem.



Optimization equation:

$P = 2x + 2y$ ------------ Equation (1)

Constraint equations: $\cos\theta = \frac{x}{D}$

$x = D\cos\theta$ ---------- Equation (2)

Likewise, $\sin\theta = \frac{y}{D}$

$y = D\sin\theta$ -------- Equation (3)

Method (1). Substitute Equation (2) and Equation (3) into Equation (1) to express perimeter P, the quantity to be maximized, in terms of variable θ.

$$P = 2D\cos\theta + 2D\sin\theta$$

$$P = 2D(\cos\theta + \sin\theta) = f(\theta)$$

Differentiate P and set its derivative to zero. $\frac{dP}{d\theta} = 2D(-\sin\theta + \cos\theta)$

$$0 = 2D(-\sin\theta + \cos\theta)$$

$$0 = -\sin\theta + \cos\theta$$

Solve the above trigonometric equation for θ.

$$0 = -\sin\,\theta + \cos\,\theta$$

$$\sin\,\theta = \cos\,\theta$$

Divide the above equation by $\cos\theta$.

$$\frac{\sin\,\theta}{\cos\,\theta} = 1$$

$$\tan\,\theta = 1$$

$$\theta = 45^{0} = \frac{\pi}{4}$$

Solve for x and y by substituting into Equation (2) and Equation (3).

$$x = D\cos\,45^{0} = \frac{\sqrt{2}}{2}D$$

$$y = D\sin\,45^{0} = \frac{\sqrt{2}}{2}D$$

Therefore the rectangle of biggest perimeter that can be cut from a circle of radius D has its breadth equals its length; or, the rectangle is a square.

Method (2). Differentiate implicitly Equation (1) with respect to one of the variables, say x, then, set $\frac{dP}{dx} = 0$ and solve for $\frac{dy}{dx}$

$$0 = 2 + 2\frac{dy}{dx}$$

$$\frac{dy}{dx} = -1 \text{ ------------- Equation (A)}$$

Differentiate Equation (2) with respect to x.

$$1 = D(-\sin\,\theta)\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{-1}{D\sin\,\theta} \text{ ------------ Equation (B)}$$

Differentiate Equation (3) with respect to x.

$$\frac{dy}{dx} = D\cos\theta\frac{d\theta}{dx} \text{ --------- Equation (C)}$$

Substitute Equation (A) and Equation (B) into Equation (C).

$$-1 = D\cos\theta\left(\frac{-1}{D\sin\theta}\right)$$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$$\theta = 45^0 = \frac{\pi}{4}$$

Therefore, we get the same results as Method 1.

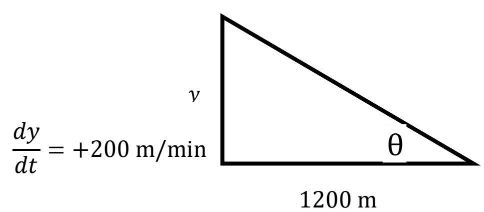
$$x = D\cos 45^0 = \frac{\sqrt{2}}{2} \text{ and } y = D\sin 45^0 = \frac{\sqrt{2}}{2}$$

Time-Rates Involving Trigonometric Function

Example 39. A balloon on the ground 1200 meters from an observer rises straight up at the rate of 200 meters per minute. How fast is the angle of elevation of the balloon from the observer's sight increasing when the balloon is at an altitude of 1600 meters?

Let: y be the height of the balloon at anytime t

θ be the angle of elevation of the balloon at anytime t



Solution: The unknown on this problem is the time-rate of θ which is $\frac{d\theta}{dt}$, when y = 1600 meters. The given condition is the constant distance of the observer, from the balloon that is rising vertically.

The first thing to do is to find the equation that relates y and θ. From the figure,

$$\tan\theta = \frac{y}{1200}$$

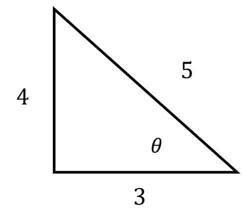
$$y = 1200\ tan\,\theta \text{ ------------- Equation (1)}$$

Differentiate Equation (1) with respect to time t.

$$\frac{dy}{dt} = 1200\sec^2\theta\frac{d\theta}{dt} \text{ ------- Equation (2)}$$

To find the value of $\sec\theta$ when $y = 1600$, from the right triangle,

$$\tan\theta = \frac{1600}{1200} = \frac{4}{3}$$



Hence, $\sec\theta = \frac{5}{3}$

Do the necessary substitutions into Equation (2):

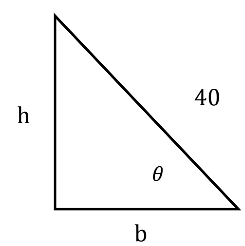
$$200 = 1200\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt}$$

$$200 = 1200\left(\frac{25}{9}\right)\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = 200\left[\frac{9}{1200\,(25)}\right] = \frac{2(9)}{12(25)} = \frac{2(3)}{4(25)} = \frac{3}{2(25)} = \frac{3}{50}\,radian/\min$$

Example 40. The measure of one of the acute angles of a right triangle is decreasing at the rate of $\frac{1}{36}\pi\ rad/\sec$. If the length of the hypotenuse remains a constant at 40 cm, find how fast the area is changing when the measure of the acute angle is $\frac{1}{6}\pi$.

Solution: The given condition is that the hypotenuse is constant at a value of 40 cm. We wish to know the time-rate of the area of the triangle $\frac{dA}{dt}$ at that instant the acute angle which is decreasing with time is $\frac{1}{6}\pi$.



Recall: $A = \frac{1}{2}bh$ --------------- Equation (1)

However from the right triangle: $\sin\theta = \frac{h}{40}$ or $h = 40\sin\theta$ -------------- Equation (2)

Similarly, $\cos\theta = \frac{b}{40}$ or $b = 40\cos\theta$ --------------- Equation (3)

Substitute Equations (2) and (3) in Equation (1).

$$A = \frac{1}{2}(40\sin\theta)(40\cos\theta) = 400\,(2\sin\theta\cos\theta)$$ --------- Equation (4)

Recall that $\sin 2\theta = 2\sin\theta\cos\theta$. Substitution into Equation (4) yields

$$A = 400\sin 2\theta$$

Differentiate the above equation with respect to t.

$$\frac{dA}{dt} = 400\cos 2\theta\left(2\frac{d\theta}{dt}\right) = 800\cos 2\theta\frac{d\theta}{dt}$$

Substitute the given information. $\frac{dA}{dt} = 800\left[\cos 2\left(\frac{\pi}{6}\right)\right]\left(-\frac{1}{36}\pi\right)$

Note that $\frac{d\theta}{dt}$ is given a negative sign since the acute angle is decreasing with time.

Therefore, $\frac{dA}{dt} = -\frac{800\pi}{36}\cos\frac{\pi}{3} = -\frac{800\pi}{36}\left(\frac{1}{2}\right) = -\frac{100}{9}\pi cm^2/\sec$

B. Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

1. $\frac{d}{dx}Arc\sin u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	4. $\frac{d}{dx}Arc\cot u = -\frac{1}{1+u^2}\frac{du}{dx}$
2. $\frac{d}{dx}Arc\cos u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	5. $\frac{d}{dx}Arc\sec u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$
3. $\frac{d}{dx}Arc\tan u = \frac{1}{1+u^2}\frac{du}{dx}$	6. $\frac{d}{dx}Arc\csc u = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

To recall, the table below shows the principal value of the inverse trigonometric functions in their domain and the corresponding range.

Function	Domain	Range
$Arc\sin x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$Arc\cos x$	$[-1, 1]$	$[0, \pi]$
$Arc\tan$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$Arc\cot x$	$(-\infty, \infty)$	$(0, \pi)$
$Arc\sec x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$Arc\csc x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

In Calculus, $Arc\sin x$, $Arc\cos x$ and $Arc\tan$ are the most important inverse trigonometric functions. However, there is often disagreement in the choice of principal value for inverse secant and inverse cosecant. Some authors define the value as between $-\pi$ and $-\frac{\pi}{2}$ for negative value of trigonometric functions secant and cosecant. Relative to this and in as much that they are seldom used, these inverse trigonometric functions may be conveniently avoided.

Example 41. Find $\frac{dy}{dx}$ using the indicated value of x, if given.

a) $y = Arc\sin(2-3x)$

On the formula $\frac{d}{dx} Arc\sin u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $u = 2-3x$ and $\frac{du}{dx} = -3$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2-3x)^2}}(-3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(4-12x+9x^2)}}(-3)$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{-3+12x-9x^2}}$$

b) $y = Arc\sec\sqrt{1+2x}$

On the formula $\frac{d}{dx} Arc\sec u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$, $u = \sqrt{1+2x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{1+2x}}(2) = \frac{1}{\sqrt{1+2x}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+2x}\sqrt{\left(\sqrt{1+2x}\right)^2-1}}\left(\frac{1}{\sqrt{1+2x}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+2x}\sqrt{(1+2x)-1}}\left(\frac{1}{\sqrt{1+2x}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\sqrt{1+2x}\right)^2\sqrt{2x}}$$

$$\frac{dy}{dx} = \frac{1}{(1+2x)\sqrt{2x}}$$

c) $y = Arc\tan\left(\frac{2-x}{2+x}\right)$

On the formula $\frac{d}{dx} Arc\tan u = \frac{1}{1+u^2}\frac{du}{dx}$,

$$u = \frac{2-x}{2+x}$$

$$\frac{du}{dx} = \frac{(2+x)(-1)-(2-x)(1)}{(2+x)^2} = \frac{-2-x-2+x}{(2+x)^2} = \frac{-4}{(2+x)^2}$$

Therefore, $\frac{dy}{dx} = \frac{1}{1+\left(\frac{2-x}{2+x}\right)^2}\left[\frac{-4}{(2+x)^2}\right] = \frac{1}{1+\frac{(2-x)^2}{(2+x)^2}}\left(\frac{-4}{(2+x)^2}\right)$

$$\frac{dy}{dx} = \frac{\cancel{(2+x)^2}}{(2+x)^2+(2-x)^2}\left[\frac{-4}{\cancel{(2+x)^2}}\right]$$

$$\frac{dy}{dx} = \frac{-4}{(4+4x+x^2)+(4-4x+x^2)}$$

$$\frac{dy}{dx} = \frac{-4}{4+4x+x^2+4-4x+x^2}$$

$$\frac{dy}{dx} = \frac{-4}{8+2x^2}$$

$$\frac{dy}{dx} = \frac{-4}{2(4+x^2)}$$

$$\frac{dy}{dx} = \frac{-2}{4+x^2}$$

d) $y = x\,Arc\cos x$; when $x = \frac{\sqrt{3}}{2}$

Use the product formula where: $u = x,\ \frac{du}{dx} = 1$

$$v = Arc\cos x, \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}(1) = -\frac{1}{\sqrt{1-x^2}}$$

Substitute on the product formula.

$$\frac{dy}{dx} = x\left[\frac{-1}{\sqrt{1-x^2}}\right] + (Arc\cos x)(1)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} + Arc\cos x$$

When : $x = \frac{\sqrt{3}}{2}$ $\qquad \frac{dy}{dx} = \frac{-\frac{\sqrt{3}}{2}}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}} + \frac{\pi}{6} = \frac{-\frac{\sqrt{3}}{2}}{\sqrt{1-\frac{3}{4}}} + \frac{\pi}{6} = \frac{-\frac{\sqrt{3}}{2}}{\sqrt{\frac{1}{4}}} + \frac{\pi}{6}$

$$\frac{dy}{dx}=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}+\frac{\pi}{6}=-\sqrt{3}+\frac{\pi}{6}$$

e) $y = x Arc\tan\frac{1}{2}x$

Use the product formula where: $u = x, \frac{du}{dx}=1$

$$v = Arc\tan\frac{1}{2}x, \frac{dv}{dx}=\frac{1}{1+\left(\frac{1}{2}x\right)^2}\left(\frac{1}{2}\right)$$

$$\frac{dv}{dx}=\frac{1}{\frac{4+x^2}{4}}\left(\frac{1}{2}\right)=\frac{4}{4+x^2}\left(\frac{1}{2}\right)=\frac{2}{4+x^2}$$

Substitute on the product formula.

$$\frac{dy}{dx}=x\left(\frac{4}{4+x^2}\right)+(Arc\tan x)(1)$$

$$\frac{dy}{dx}=(x)\left(\frac{2}{4+x^2}\right)+\left(Arc\tan\frac{1}{2}\right)x$$

$$\frac{dy}{dx}=\frac{2x}{4+x^2}+Arc\tan\frac{1}{2}x$$

f) $Arc\tan\frac{x}{y}=x-y$

Use implicit differentiation.

$$\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)=1-y'$$

$$\frac{1}{\frac{y^2+x^2}{y^2}}\left[\frac{y-xy'}{y^2}\right]=1-y'$$

$$\frac{y-xy'}{y^2+x^2}=1-y'$$

$$y-xy'=y^2+x^2-(y^2+x^2)y$$

$$y'(y^2+x^2-x)=y^2+x^2-y$$

$$y'=\frac{y^2+x^2-y}{y^2+x^2-x}$$

Example 42. Find the equation of the tangent line to the curve $y=\frac{1}{x}Arc\tan\frac{1}{x}$ at point $\left(1.\frac{\pi}{4}\right)$.

Solution: Slope of tangent line at any point on the curve is given by $\frac{dy}{dx}$ which is obtained using the product formula where:

$$u=\frac{1}{x},\frac{du}{dx}=-\frac{1}{x^2}$$

$$v=Arc\tan\frac{1}{x},\frac{dv}{dx}=\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)=-\frac{1}{x^2}\left(\frac{x^2}{1+x^2}\right)=-\frac{1}{1+x^2}$$

Substitute on the product formula,

$$\frac{dy}{dx}=\frac{1}{x}\left(-\frac{1}{1+x^2}\right)+\left[Arc\tan\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right)$$

$$\frac{dy}{dx}=-\frac{1}{x}\left(\frac{1}{1+x^2}\right)+\left[Arc\tan\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right)$$

$$\frac{dy}{dx}=-\frac{1}{1}\left(\frac{1}{1+1}\right)-\frac{1}{1}Arc\tan 1=-\frac{1}{2}-\frac{\pi}{4}=\frac{-2-\pi}{4}=-\frac{2+\pi}{4}$$

Use the point-slope form to get equation of the tangent line.

$$y-\frac{\pi}{4}=\frac{-2-\pi}{4}(x-1)$$

$$\frac{4y-\pi}{4}=\frac{-(2+\pi)x+2+\pi}{4}$$

$$4y-\pi+(2+\pi)x-2-\pi=0$$

$$4y+(2+\pi)x-2-2\pi=0$$

$$4y+(2+\pi)x-2(1+\pi)=0$$

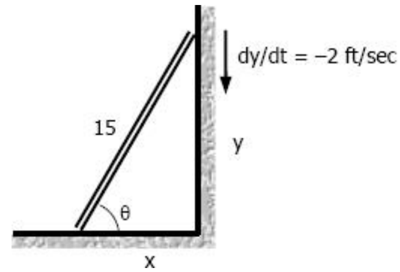
Example 43. A ladder 15 feet long leans against a vertical wall. If the top slides down at 2 feet/sec, how fast is the angle of elevation of the top of the ladder decreasing as observed from its foot, when the lower end is 12 feet from the wall? Use Inverse Trigonometric Function.

Solution: Let x be the distance of the foot of the ladder from the wall at any time t

y be the distance of the top of the ladder from the ground at any time t.

θ be the angle of elevation of the top of the ladder as observed from its foot at any time t.

Solution: First, find the equation that relates the variables θ and y. We use the definition of either sin θ or csc θ.



Let us choose csc θ.

$$\csc\theta = \frac{15}{y}$$

$$\theta = Arc\csc\frac{15}{y}$$

Differentiate the above equation with respect to t.

$$\frac{d\theta}{dt} = -\frac{1}{\left(\frac{15}{y}\right)\sqrt{\left(\frac{15}{y}\right)^2 - 1}}\frac{d}{dt}\left(\frac{15}{y}\right)$$

$$\frac{d\theta}{dt} = -\frac{1}{\left(\frac{15}{y}\right)\sqrt{\frac{225 - y^2}{y^2}}}\left[-\frac{15}{y^2}\left(\frac{dy}{dt}\right)\right]$$

$$\frac{d\theta}{dt} = -\frac{1}{\frac{15}{y^2}\sqrt{225 - y^2}}\left[-\frac{15}{y^2}(-2)\right]$$

$$\frac{d\theta}{dt} = \frac{y^2}{\sqrt{25 - y^2}}\left(-\frac{2}{y^2}\right) = \frac{-2}{\sqrt{25 - y^2}}$$

When $x = 12,\ y = \sqrt{(15)^2 - (12)^2} = \sqrt{225 - 144} = \sqrt{81 = 9}$

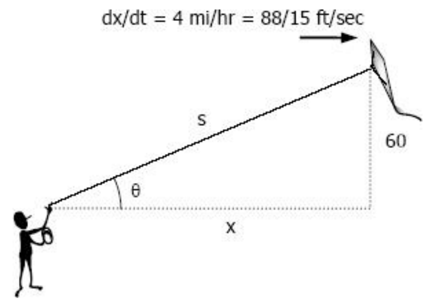
Therefore, $\frac{d\theta}{dt} = \frac{-2}{\sqrt{225 - 81}} = \frac{-2}{\sqrt{144}} = \frac{-2}{12} = -\frac{1 rad}{6\sec}$

The negative sign of $\frac{d\theta}{dt}$, the time rate of the angle of elevation of the top of the ladder signifies that as the top of the ladder slides down the wall, θ is decreasing with time.

Example 44. A kite is 60 feet high with 100 feet of cord out. If the kite is moving horizontally 4 mi/hr directly away from the boy flying it, find the rate of change of the angle of elevation of the cord. Solve using an inverse trigonometric function.

Solution: The variables in the problem are the length of the cord s that is out; the horizontal distance of the kite x from the boy and the angle of elevation of the cord θ at anytime t. The height of the kite is fixed at 60 feet.

The equation relating the variables is, as seen on the figure,



$$\tan\theta = \frac{60}{x}$$

$$\theta = Arc\tan\left(\frac{60}{x}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{60}{x}\right)^2}\left(-\frac{60}{x^2}\right)\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\frac{x^2+3600}{x^2}}\left(-\frac{60}{x^2}\right)\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{60}{x^2+3600}\left(\frac{dx}{dt}\right)$$

When $s = 100\ ft, x = \sqrt{s^2 - (60)^2} = \sqrt{(100)^2 - (60)^2}$

$$x = \sqrt{10{,}000 - 3600} = \sqrt{6400} = 80\ feet$$

$$\frac{d\theta}{dt} = -\frac{60}{(80)^2+3600}\left(\frac{88}{15}\right) = \frac{-4(88)}{6400+3600} = \frac{-4(88)}{10{,}000} = \frac{-88}{2{,}500} = \frac{-22\,rad}{625\sec}$$

The negative sign of $\frac{d\theta}{dt}$ implies that as the cord of the kite lengthens, angle θ is decreasing with time.

C. Derivatives of Exponential and Logarithmic Functions

In your study of General Mathematics, you learned that the inverse of an exponential function is the logarithmic function, and vice-versa.

For the exponential function $f(x) = b^x$, where $f(x) > 0$, the base $b > 0,\ b \neq 1$. Its inverse function is $\log_b f(x) = x$, where $f(x)$ is referred to as the argument of the logarithm and b is the base. This leads to the definition that the logarithm of the argument $f(x)$ is the exponent x to which the base b must be raised to produce the argument $f(x)$.

Let u be a differentiable function of $x, a > 0, a \neq 1$. The following formulas are used to differentiate logarithmic and exponential functions.

1) $\frac{d}{dx}e^u = e^u \frac{du}{dx}$	3) $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
2) $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$	4) $\frac{d}{dx}\log_a u = \frac{1}{u}\log_a e \frac{du}{dx}$

Listed below are properties of logarithms and exponential functions that may help students when differentiating the function.

Properties of Logarithms

1) $\log_b 1 = 0$	5) $\log_b M^n = n \log_b M$
2) $\log_b b = 1$	6) $\log_b \sqrt[n]{M} = \frac{1}{n}\log_b M$
3) $\log_b MN = \log_b M + \log_b N$	7) $b^{\log_b M} = M$
4) $\log_b \frac{M}{N} = \log_b M - \log_b N$	8) $\log_b M = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}$

Example 45. Find $\frac{dy}{dx}$ and simplify, whenever possible.

a) $y = e^{8-4x^2}$

Use the formula $\frac{d}{dx}e^u = e^u \frac{du}{dx}$, *where* $u = 8 - 4x^2$, $\frac{du}{dx} = -8x$.

Hence, $\frac{dy}{dx} = e^{8-4x^2}(-8x) = -8xe^{8-4x^2}$

b) $y = \log \sqrt{2x-7}$

Use the formula : $\log_a u = \frac{1}{u}\log_a e \frac{du}{dx}$, *where* $u = \sqrt{2x-7}$, $\frac{du}{dx} = \frac{1}{2\sqrt{2x-7}}(2) = \frac{1}{\sqrt{2x-7}}$

Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{2x-7}}\log e\left(\frac{1}{\sqrt{2x-7}}\right) = \frac{\log e}{\left(\sqrt{2x-7}\right)^2} = \frac{\log e}{2x-7}$

c) $y = \ln \cos x$

Use the formula $\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}$ where $u = \cos x$, $\frac{du}{dx} = (-\sin x)(1) = -\sin x$ $u = \cos x$

Hence, $\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$

d) $y = 5^{-3x}$

Use the formula $\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$ where $u = -3x$, $\frac{du}{dx} = -3$

Therefore, $\frac{dy}{dx} = 5^{-3x} \ln 5(-3) = -3(\ln 5)(5^{-3x})$

e) $y = x^3(3\ln x - 1)$

Use the product formula where: $u = x^3$, $\frac{du}{dx} = 3x^2$

$$v = 3\ln x - 1,\ 3\left(\frac{1}{x}\right)(1) = \frac{3}{x}$$

Therefore, $\frac{dy}{dx} = x^3\left(\frac{3}{x}\right) + (3\ln x - 1)(3x^2)$

$$\frac{dy}{dx} = 3x^2 + 3x^2(3\ln x - 1)$$

Factor. $\frac{dy}{dx} = 3x^2(1 + 3\ln x - 1) = 3x^2(3\ln x) = 9x^2 \ln x$

f) $y = \frac{1}{2}x^2 e^{-2x}$

Use the product formula where $u = x^2$, $\frac{du}{dx} = 2x$, $v = e^{-2x}$, $\frac{dv}{dx} = e^{-2x}(-2) = -2e^{-2x}$

Substitute. $\frac{dy}{dx} = \frac{1}{2}\left[x^2(-2e^{-2x}) + e^{-2x}(2x)\right]$

$$\frac{dy}{dx} = \frac{1}{2}\left[2xe^{-2x}(-x+1)\right] = xe^{-2x}(1-x)$$

g) $y = (1 - e^{-2x})^3$

Use the general power formula where $n = 3$, $u = 1 - e^{-2x}$, $\frac{du}{dx} = -e^{-2x}(-2) = 2e^{-2x}$

Hence, $\frac{dy}{dx} = 3(1 - e^{-2x})^2(2e^{-2x}) = 6e^{-2x}(1 - e^{-2x})^2$

h) $y = \ln \dfrac{1+e^{-2x}}{1-e^{-2x}}$

Method 1. Use the quotient formula where $u = 1+e^{-2x}$, $\dfrac{du}{dx} = -e^{-2x}(-2) = -2e^{-2x}$

$v = 1-e^{-2x}$, $\dfrac{dv}{dx} = -e^{-2x}(-2) = 2e^{-2x}$

Therefore, $$\frac{dy}{dx} = \frac{1}{\dfrac{1+e^{-2x}}{1-e^{-2x}}}\left[\frac{(1-e^{-2x})(-2e^{-2x})-(1+e^{-2x})(2e^{-2x})}{(1-e^{-2x})^2}\right]$$

$$\frac{dy}{dx} = \frac{1-e^{-2x}}{1+e^{-2x}}\left[\frac{2e^{-2x}(-1+e^{-2x}-1-e^{-2x})}{(1-e^{-2x})^2}\right]$$

$$\frac{dy}{dx} = \frac{2e^{-2x}(-2)}{(1+e^{-2x})(1-e^{-2x})} = \frac{-4e^{-2x}}{(1+e^{-2x})(1-e^{-2x})}$$

The above $\dfrac{dy}{dx}$ can further be simplified by getting the product of $(1+e^{-2x})(1-e^{-2x}) = 1-e^{-4x}$.

Hence, $\dfrac{dy}{dx} = \dfrac{-4e^{-2x}}{1-e^{-4x}}$.

Method 2. We first simplify $\ln \dfrac{1+e^{-2x}}{1-e^{-2x}}$ using the property of logarithm $\log_b \dfrac{M}{n} = \log_b M - \log_b N$. Thus, $y = \ln(1+e^{-2x}) - \ln(1-e^{-2x})$.

$$\frac{dy}{dx} = \frac{1}{1+e^{-2x}}(e^{-2x})(-2) - \frac{1}{1-e^{-2x}}(e^{-2x})(-2)$$

Factor. $$\frac{dy}{dx} = -2e^{-2x}\left[\left(\frac{1}{1+e^{-2x}}\right)+\left(\frac{1}{1-e^{-2x}}\right)\right] = -2e^{-2x}\left[\frac{1-e^{-2x}+1+e^{-2x}}{(1+e^{-2x})(1-e^{-2x})}\right]$$

$$\frac{dy}{dx} = -2e^{-2x}\left[\frac{2}{1-e^{-4x}}\right] = \frac{-4e^{-2x}}{1-e^{-4x}}$$

i) $y = Arc\tan \ln x$

Use the formula $\dfrac{d}{dx} Arc\tan u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, where $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}(1) = \dfrac{1}{x}$

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2}\left(\dfrac{1}{x}\right) = \dfrac{1}{x(1+\ln^2 x)}$.

j) $y = \ln^2 \sin x$

The given can be expressed as $y = (\ln \sin x)^2$. Now, use the general power formula

where $n = 2,\ u = \ln \sin x,\ \dfrac{du}{dx} = \dfrac{1}{\sin x}(\cos x) = \dfrac{\cos x}{\sin x} = \cot x$.

Therefore, $\dfrac{dy}{dx} = 2(\ln \sin x)(\cot x) = 2\cot x(\ln \sin x)$

j) $y = \ln\left(1 + e^{-3x}\right)^2$

Method 1. Use the formula $\dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}$

$$where,\ u = \left(1 + e^{-3x}\right)^2, \frac{du}{dx} = 2\left(1 + e^{-3x}\right)\left[\left(e^{-3x}\right)(-3)\right] = -6e^{-3x}\left(1 + 3^{-3x}\right)$$

$$Therefore,\ \frac{dy}{dx} = \frac{1}{\left(1 + e^{-3x}\right)^2}\left[-6e^{=3x}\left(1 + e^{-3x}\right)\right] = \frac{-6e^{-3x}}{1 + e^{-3x}}$$

Method 2. Simplify $y = \ln\left(1 + e^{-3x}\right)^2$ using the property of logarithm

$$\log_b M^b = n \log_b M$$

Hence, $y = 2\ln\left(1 + e^{-3x}\right)$

$$\frac{dy}{dx} = 2\left[\frac{1}{1 + e^{-3x}}\left(e^{-3x}\right)(-3)\right] = \frac{-6e^{-3x}}{1 + e^{-3x}}$$

k) $e^x + e^y = e^{xy}$

Differentiate implicitly. $e^x(1) + e^y = e^{xy}\left(x\dfrac{dy}{dx} + y\right)$

$$e^x + e^y y' = xe^{xy}y' + ye^{xy}$$

$$e^y y' - xe^{xy}y' = ye^{xy} - e^x$$

Example 46. Given that $ln\left(\frac{y}{2x-1}\right) = 0$ find the equation of the tangent line when $x = 1$.

Solution: Method 1. We first express y as a function of x by transforming the given logarithmic equation to its equivalent exponential equation.

$$\ln\left(\frac{y}{2x - 1}\right) = 0$$

$$e^0 = \frac{y}{2x - 1}$$

However, $e^0 = 1$. $1 = \dfrac{y}{2x-1}$

$y = 2x - 1$ ----------- Equation (A)

Differentiate. $\dfrac{dy}{dx} = 2$

Hence, slope of the tangent line equals 2.
Find the corresponding value of y when $x = 1$. Substitute into Equation (A).

$$y = 2(1) - 1 = 1$$

Equation of tangent line:

$$y - 1 = 2(x - 1)$$
$$y - 1 = 2x - 2$$
$$2x - y - 1 = 0$$

Method 2. There is another way to arrive to Equation (A). Use the property of logarithm $\log_b \dfrac{M}{n} = \log_b M - \log_b N$.

Hence,

$$\ln\left(\frac{y}{2x-1}\right) = 0$$
$$\ln y - \ln(2x-1) = 0$$
$$\ln y = \ln(2x-1)$$

However, if $\ln A = \ln B$, then, $A = B$.
Hence, $y = 2x - 1$

Observe that we arrive at Equation (A) too but using a different method. Expectedly, we will get the same equation of the tangent line at point where $x = 1$.

Example 47. Find and classify the critical points of the curve $y = x\ln x$
Solution: Differentiate using the product formula.

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

At the critical point of the curve, $\dfrac{dy}{dx} = 0$.

Hence, $0 = 1 + \ln x$

$$\ln x = -1$$

Transform the above logarithmic equation to its equivalent exponential equation.

$$e^{-1} = x$$
$$x = \frac{1}{e} = 0.37$$

Solve the corresponding value of y.

$$y=\frac{1}{e}\ln e^{-1}=\frac{1}{e}(-1\ln e)=\frac{1}{e}(-1)=-\frac{1}{e}=-0.37$$

Find the second derivative and use it to classify the critical point $\left(\frac{1}{e},-\frac{1}{e}\right)$.

$$y''(x)=\frac{d}{dx}(1+\ln x)=\frac{1}{x}$$

$$y''(0.37)=\frac{1}{0.37}$$

Since $y''(0.37)>0$, then, the critical point $\left(\frac{1}{e},-\frac{1}{e}\right)$ is a minimum point.

Example 48. Show that the curve $y=xe^{-\frac{x}{2}}$ has its maximum point at $\left(2,\frac{2}{e}\right)$.

Solution: Differentiate using the product formula.

$$\frac{dy}{dx}=x\left(e^{-\frac{x}{2}}\right)\left(-\frac{1}{2}\right)+e^{-\frac{x}{2}}(1)=e^{-\frac{x}{s}}\left(-\frac{x}{2}+1\right)$$

$$\frac{dy}{dx}=\frac{1}{2}e^{-\frac{x}{2}}(-x+2)$$

Equate each factor to zero.

$$-x+2=0 \quad \text{or} \quad e^{-\frac{x}{2}}=0$$

$$x=2 \quad \text{or} \quad \ln e^{-\frac{x}{2}}=\ln\ =0$$ (Rejected since $\ln 0$ has no value)

When, $x=2,\ y=2e^{-\frac{2}{2}}=2e^{-1}=\frac{2}{e}$

Perform the second derivative test. $y''(x)=e^{-\frac{x}{2}}\left(-\frac{1}{2}\right)+\left(\frac{-x+2}{2}\right)\left(e^{-\frac{x}{2}}\right)\left(-\frac{1}{2}\right)$

$$y''(x)=-\frac{1}{2}e^{-\frac{x}{2}}\left[1+\left(\frac{-x+2}{2}\right)\right]=-\frac{1}{2}e^{-\frac{x}{2}}\left]\left(\frac{2-x+2}{2}\right)\right.$$

$$y''(x)=-\frac{1}{4}e^{-\frac{x}{2}}(4-x)$$

When $x=2.\ y''(2)=-(e^{-1})(2)=-e^{-1}=-\frac{1}{e}=-0.37$

Since $y''(2)<0$, therefore, point $\left(2,\frac{2}{e}\right)$ is a relative maximum point of the given curve $y=xe^{-\frac{x}{2}}$.

Example 49. If $y=\ln\frac{1+x}{1-x}$ and if the time-rate of y is 4 unit/seconds, find the time-rate of x when $y=\ln 3$.

Solution: Using the appropriate property of logarithm, simplify $y=\ln\frac{1+x}{1-x}$.

$$y=\ln(1+x)-\ln(1-x)$$

Differentiate with respect to t.

$$\frac{dy}{dt}=\frac{1}{1+x}\frac{dx}{dt}-\frac{1}{1-x}\frac{dx}{dt}$$

When $y=\ln 3$, we find now the corresponding x-value by substitution on the given equation $y=\ln\frac{1+x}{1-x}$.

$$\ln 3=\ln\frac{1+x}{1-x}$$

$$3=\frac{1+x}{1-x}$$

$$3(1-x)=1+x$$

$$3-3x=1+x$$

$$4x=2$$

$$x=\frac{1}{2}$$

Therefore, when $x=\frac{1}{2}$ and $\frac{dy}{dt}=4$,

$$4=\frac{1}{1+\frac{1}{2}}\left(\frac{dx}{dx}\right)-\frac{1}{1-\frac{1}{2}}\left(\frac{dx}{dt}\right)$$

$$4=\frac{2}{3}\left(\frac{dx}{dt}\right)-2\left(\frac{dx}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{2}{3}-2\right)$$

$$4=\left(\frac{dx}{dt}\right)\left(\frac{2}{3}+\frac{4}{3}\right)=\frac{8}{3}\frac{dx}{dt}$$

$$\frac{dx}{dt}=\frac{12}{8}=\frac{3}{2}\,units/\sec ond$$

Activity Sheet

FOUR-STEP RULE/ INCREMENT METHOD

Name : ______________________________ **SCORE :** __________

Section : ____________________ **Class Schedule :** ____________________

Directions: Find $\frac{dy}{dx}$ using the Four-Step Rule/Increment Method of Differentiation.

1. $y = 3x^2 - 2x + 5$

2. $y = x^3 - 4x$

3. $y = \frac{5}{2 - x}$

Activity Sheet

FOUR-STEP RULE/ INCREMENT METHOD

Name : ______________________________ **SCORE :** __________

Section :______________________**Class Schedule :** ________________

Directions: In each of the following, find $\frac{dy}{dx}$ when x = 3 using the Increment Method.

1. $y = x(x+1)$

2. $y = \frac{4}{\sqrt{x-1}}$

3. $y = \frac{1}{4x^3}$

Activity Sheet
DIFFERENTIATION RULES

Name : ____ **SCORE :** ___________

Section :_____________________________ **Class Schedule :** _____________________

Directions: Apply the appropriate differentiation rules to find the derivatives of the given function.

1. $f(x) = 5x^2$

2. $f(x) = x^4 - x^{-3}$

3. $f(x) = 7x^3 - 3x^2 + 9$

4. $f(x) = (5x - 7)^3$

5. $f(x) = 9x^4 + 6\sqrt[3]{x} + 7$

Activity Sheet
PRODUCT RULE

Name : ______________________ **SCORE :** __________

Section : ____________________ **Class Schedule :** ______________

Directions: Use Product Rule to differentiate each function. Simplify your answer

1. $f(x) = 4x(x-5)$

2. $f(x) = 3x^4(2x^3-6)$

3. $f(x) = (3x-4)(x-7)$

4. $f(x) = (x^2+2x)(x^3-4x)$

5. $f(x) = (5x^6-2x^2)(2x^2+3x)$

Activity Sheet
QOUTIENT RULE

Name : ______________________________ **SCORE :** __________

Section :______________________**Class Schedule :** ________________

Directions: Use Quotient Rule to differentiate each function. Simplify your answer

1. $f(x) = \dfrac{x}{x+6}$

2. $f(x) = \dfrac{4x+7}{x-2}$

3. $f(x) = \dfrac{2x^3}{4x^2+5}$

4. $f(x) = \dfrac{4x^5+3x}{4-2x^3}$

5. $f(x) = \dfrac{2x^4-4x^3+8}{x^3+2x-1}$

Activity Sheet
DIFFERENTIATION RULES

Name : ______________________ **SCORE :** __________

Section :__________________ **Class Schedule :** ______________

Directions: Differentiate each function. Do not expand any expression before differentiating

1. $f(x) = (3x^2 + 2x - 5)^4$

2. $f(x) = (7 + 2x - x^3 + x^4)^6$

3. $f(x) = (5x - 9)^{\frac{3}{4}}$

4. $f(x) = \dfrac{5}{\left(x^3 - 2x + 1\right)^2}$

5. $f(x) = \dfrac{x - 3}{\sqrt{x^3 - 8}}$

Activity Sheet
PRODUCT RULE AND POWER OF A FUNCTION RULE

Name : ______________________________ **SCORE :** __________

Section :______________________**Class Schedule :** ________________

Directions: Use the Product Rule and the Power of a Function Rule to differentiate the following. Simplify.

1. $f(x) = (4x+2)^3(2x+1)$

2. $f(x) = (8-3x^2)(4x-1)^6$

3. $f(x) = (7x-3)^4(x^3+1)^3$

4. $f(x) = (6-2x^2)\sqrt{5x^3+4}$

5. $f(x) = \sqrt[4]{2x-5}(7x^2-3)^5$

Activity Sheet
HIGHER DERIVATIVES

Name : ______________________________ **SCORE :** __________

Section :____________________**Class Schedule :** ______________

Directions: Find the indicated derivative.

1. $f(x) = 5x^5 - 2x^4 + x^3 - 21$. Find $f''(x)$.

2. $f(x) = 3x^2 - \frac{5}{x^4}$. Find $\frac{d^3y}{dx^3}$.

3. $f(x) = \sqrt[3]{2x-3}$. Find $f^{(4)}(x)$.

4. $f(x) = (2x-1)(3x^2+2)$. Find $\frac{d^2y}{dx^2}$.

5. $f(x) = \frac{4}{x+1}$. Find $f'''(x)$.

Activity Sheet
DERIVATIVE AS SLOPE OF THE TANGENT LINE

Name : ______________________________________ **SCORE :** ____________

Section :_________________________**Class Schedule :** ___________________

I. Find the equation of the tangent line to the given curves.

1. $y = \sqrt{x^2 - 25}$ at the point $(13,12)$.

2. $y = \sqrt[3]{7x - 6}$ that is perpendicular to the line $12x + 7y + 2 = 0$

II. Find an equation of the normal line to the curve

1. $y = (x^2 - 1)^2$ at the point $(-2,9)$.

2. $y = x^3 - 4x$ and that is parallel to the line $x + 8y - 8 = 0$

Activity Sheet
IMPLICIT DIFFERENTIATION

Name : ______________________________ **SCORE :** __________

Section :____________________**Class Schedule :** ______________

Directions: Determine $\frac{dy}{dx}$ using implicit differentiation.

1. $x^2 + y^2 = 45$

2. $27y^2 = 4x^4$

3. $x^3 + y^3 = xy$

4. $(x + y)(x - y) = 7$

5. $(3xy + 1)^5 = x^3$

Activity Sheet
IMPLICIT DIFFERENTIATION

Name : ______________________________ **SCORE :** __________

Section :______________________ **Class Schedule :** ______________

Directions: Determine the slope of the curve at the given point. Furthermore, find the equation of the tangent line at the given point.

1. $x^2 + y^2 - 12x + 4y - 5 = 0$ at (0, 1)

2. $(x + 2y)^2 = x + 10$ at (−1, 2)

3. $(3x - y)^2 = 6x + 2y + 23$ at (1, -2)

4. $y(3x - y^2) = 16$ at (4, 2)

5. $x(x^2 - y^2) = 3$ at (-1, 2)

Activity Sheet
CHAIN RULE

Name : ______________________________ **SCORE :** __________

Section :______________________ **Class Schedule :** ______________

Directions: Find the derivative of each of the following functions.

1. $y = \left(x^2 + 4x + 6\right)^5$

2. $y = \left(x^3 - 5x\right)^4$

3. $y = \left(2x^2 - 6x + 1\right)^{-8}$

4. $y = \sqrt{x^2 - 7x}$

5. $f(x) = \left(x - \frac{1}{x}\right)^{\frac{3}{2}}$

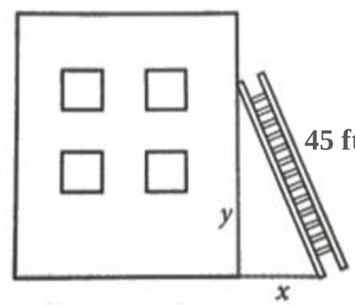
Activity Sheet
APPLICATION OF DIFFERENTIATION

Name : ______________________________ **SCORE :** __________

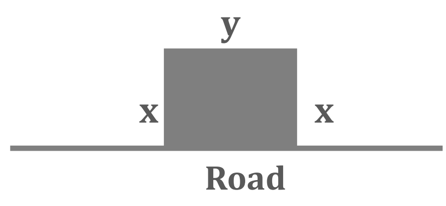
Section :__________________ **Class Schedule :** ______________

Directions: Solve each of the following problems

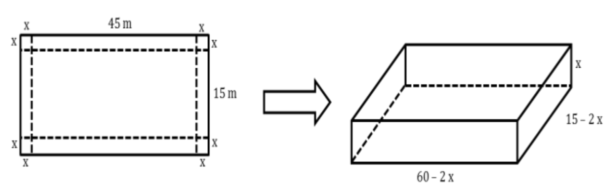
1. A 45 – foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 3 feet per second, at what rate is the top of the ladder sliding down the wall when the top of the ladder is 30 feet above the floor?



2. A farmer has 1, 000 m of fencing material and wishes to enclose a rectangular field. One side of the field is against the road, which is already fenced, so the farmer needs to fence the remaining three sides of the rectangular field only. The farmer wishes to use all of the fencing material. Find the dimensions that achieve this.



3. A piece of card board 45 m by 15 m is to be used to make a rectangular box with an open top. Find the dimensions that will give the box with the largest volume.



Activity Sheet
DERIVATIVE OF EXPONENTIAL, LOGARITHMIC AND TRIGONOMETRIC FUNCTIONS

Name : ________________________________ **SCORE :** __________

Section :____________________**Class Schedule :** ______________

A. Find the derivative of exponential function

1. $f(x) = 10^{4x-7}$

2. $f(x) = e^{2x-7}$

3. $f(x) = e^{7-8x-x^2}$

4. $f(x) = e^{x^2} + 3e^{-x}$

5. $f(x) = \dfrac{e^{-x^3}}{x}$

B. Find the derivative of logarithmic function

1. $f(x) = 3\ln x^5$

2. $f(x) = \ln\sqrt{3x^2 + 2x + 1}$

3. $f(x) = 2^x \log_2(x^4)$

4. $f(x) = \log_5(x^3 - 2x^2 + 10)$

5. $f(x) = \log(4x^3) + 4^{x^3 + x}$

C. Find the derivative of trigonometric function

1. $f(x) = \tan(4x^5)$

2. $f(x) = \sin \dfrac{1}{x}$

3. $f(x) = x^3 + \cos x + \sin \dfrac{\pi}{4}$

4. $f(x) = 4x^2 \sin 2x - 3x \cos x$

5. $f(x) = \dfrac{\sin x}{1 + \cos x}$

CHAPTER 3

ANTIDIFFERENTIATION

Learning Competencies

At the end of the chapter, the students learn to:

- Illustrate an antiderivative of a function
- Compute the general antiderivative of power of a polynomial, logarithmic, exponential, and trigonometric functions.
- Compute the antiderivative of a function using substitution rule and table of integrals (including those whose antiderivatives that involve logarithmic and inverse trigonometric functions).
- Solve separable differential equations using antidifferentiation.
- Solve exponential growth and decay, bounded growth, and logistic growth.
- Approximate the area of a region under a curve using Riemann sums: (a) left, (b) right, and (c) midpoint.
- Define the definite integral as the limit of the Riemann sums.
- Illustrate the Fundamental Theorem of Calculus.
- Compute the definite integral of a function using the Fundamental Theorem of Calculus.
- Illustrate the substitution rule.
- Compute the definite integral using the substitution rule.
- Compute the area of a plane region using the definite integral.
- Solve problems involving areas of plane regions.

3.1. Antidifferentiation

In this chapter, we develop the inverse operation of differentiation called antidifferentiation (or indefinite integration). These two operations are said to be inverse operations, just like addition and subtraction or multiplication and division, same as raising to a power and extracting root. In antidifferentiation, we find a function, say $F(x)$ whose derivative $f(x)$ or whose differential $f(x)dx$ on the certain interval of the x-axis is given. Performing antidifferentiation or indefinite integration results to $F(x)$ plus an arbitrary constant. In symbol, we write,

$$\boxed{\int f(x)dx = F(x) + C}$$

This is read "the integral of $f(x)dx$ equals $F(x)$ plus C".

Where:

- The symbol $\int$ that looks like an elongated letter "S" is called the antidifferentiation or integration sign specifying the operation of antidifferentiation (or integration) to be done upon the given derivative $f(x)$ or upon the differential $f(x)dx$. The notation dx tells us that the variable of integration is x.
- $f(x)$ is called the integrand which refers to the given derivative of the unknown function $F(x)$.
- $f(x)dx$ is the given differential.
- $F(x)$ is called the particular integral, the unknown function.
- C is the constant of integration
- $F(x)+C$ is called the indefinite integral

Theorem: Two or more functions that have the same derivative differ at most by a constant.

To illustrate what the theorem means, let us try finding the antiderivative or the indefinite integral of differential x^6dx which in symbol form is $\int x^6dx$. This task means finding a function $F(x)$ whose derivative is x^6 or looking for a function $F(x)$ having a differential of x^6dx.

Hence, $\int x^6dx$ could be any of the many functions, some of which are the following:

- $\frac{x^7}{7}$

- $\frac{x^7}{7}+3$
- $\frac{x^7}{7}-\frac{1}{4}$

In general, the antiderivative of x^6dx can be expressed in the form $\frac{x^7}{7}$ + C which is referred to as the indefinite integral of the given function, where $\frac{x^7}{7}$ is the particular integral and C is the constant of integration that differentiates one function from the others depending on given condition.

3.2. Antidifferentiation (Indefinite Integration) of Powers

The following are properties of indefinite integrals. These can be proven using differentiation.

Property 1. $\int dx = x + C$

The integral of the differential of a variable equals the variable itself.

Property 2. If a is some constant, $\int af(x)dx = a\int f(x)dx$

In other words, constant can be factored out of the integral sign.

Property 3. $\int [f(x)dx \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

The integral of a sum (or difference) of functions equals the sum (or difference) of the integrals of each function. This property be extended to any number of functions.

Property 4. $d\int f(x)dx = f(x)$

The differential of the integral of a function equals the function itself.

Property 5. $\int x^n dx = \frac{x^{n+1}}{n+1}+C$, provided $n \neq -1$.

The integral of a power of variable x, say x^n, is equal to variable x raised to exponent $(n+1)$ divided by $(n+1)$.

Example 1. Evaluate the following integrals.

a) $\int 2x^4 dx$

Solution: We first use Property 2, factor constant 2 out of the integral sign. Then, integrate using Property 5, with $n = 4$. Substitution yields

$$\int 2x^4 dx = 2\int x^4 dx = 2\left[\frac{x^{4+1}}{4+1}\right] + C = 2\left[\frac{x^5}{5}\right] + C = \frac{2}{5}x^5 + C$$

b) $\int (x-1)(x+4)dx$

Solution: Since at this point in time, we don't have a formula to integrate a product of functions, we first perform multiplication to reduce the product of functions into a sum of functions. Hence,

$$\int (x-1)(x+4)dx = \int (x^2 + 3x - 4)dx$$

Use now Property 3, followed by Property 5.

$$\int (x-1)(x+4)dx = \int (x^2 + 3x - 4)dx = \int x^2 dx + \int 3x dx + \int (-4)dx$$

Use now the appropriate property of indefinite integral.

$$\int (x-1)(x+4)dx = \frac{x^{2+1}}{2+1} + 3\int x dx - 4\int dx$$

$$= \frac{x^3}{3} + 3\left(\frac{x^2}{2}\right) - 4x + C$$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + C$$

c) $\int (4x-3)^2 dx$

Solution: Using the special product, expand the square of binomial $(4x-3)$, that is,

$(a \pm b)^2 = a^2 \pm 2ab + b^2$. Therefore,

$$\int (4x-3)^2 = \int (16x^2 - 24x + 9)dx = \int 16x^2 dx - \int 24x dx + \int 9dx$$

$$= 16\int x^2 dx - 24\int x dx + 9\int dx$$

$$= 16\left(\frac{x^3}{3}\right) - 24\left(\frac{x^2}{2}\right) + 9x + C$$

$$= \frac{16}{3}x^3 - 12x^2 + 9x + C$$

d) $\int (2 + x^2)^3 dx$

Solution: Expand cube of the binomial using the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Thus,

$$\int (2 + x^2)^3 dx = \int (8 + 12x^2 + 6x^4 + x^6) dx$$
$$= \int 8dx + \int 12x^2 dx + \int 6x^4 dx + \int x^6 dx$$
$$= 8\int dx + 12\int x^2 dx + 6\int x^4 dx + \int x^6 dx$$
$$= 8x + 12\left(\frac{x^3}{3}\right) + 6\left(\frac{x^5}{5}\right) + \frac{x^7}{7} + C$$
$$= 8x + 4x^2 + \frac{6}{5}x^5 + \frac{1}{7}x^7 + C$$

e) $\int \frac{(1-\sqrt{x})(2+\sqrt{x})}{\sqrt{x}} dx$

Solution: Reduce the given fraction into a sum of functions.

$$\int \frac{(1-\sqrt{x})(2+\sqrt{x})}{\sqrt{x}} dx = \int \frac{(2-\sqrt{x}-x)}{\sqrt{x}} dx = \int \left(\frac{2}{\sqrt{x}} - 1 - \sqrt{x}\right) dx$$

Transform radical expression to equivalent power having a fractional exponent.

$$\int \frac{(1-\sqrt{x})(2+\sqrt{x})}{\sqrt{x}} dx = \int \left(\frac{2}{x^{\frac{1}{2}}} - 1 - x^{\frac{1}{2}}\right) dx$$

Now, we use the definition of negative exponent.

$$= \int \left(2x^{-\frac{1}{2}} - 1 - x^{\frac{1}{2}}\right) dx$$
$$= 2\int x^{-\frac{1}{2}} dx - \int dx - \int x^{\frac{1}{2}} dx$$
$$= 2\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) - x - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$
$$= 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - x - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= 4x^{\frac{1}{2}} - x - \frac{2x^{\frac{3}{2}}}{3} + C$$

Transform power with fractional exponent back to equivalent radical expression.

$$= 4\sqrt{x} - x - \frac{2\sqrt{x^3}}{3} + C$$

Simplify the radical. $$= 4\sqrt{x} - x - \frac{2}{3}x\sqrt{x} + C$$

3.3. Standard Antidifferentiation (Indefinite Integration) Formulas

So far, our integration process has been limited only to the power formula where the base is a variable. To speed-up the work of evaluating integrals, there are integration formulas for certain standard forms. Carrying-out integration requires comparing these forms and if found identical with or is reducible to one of them by applying valid transformation, the corresponding formula is then used.

We need to remember that the test for correctness of the result of integration is that the derivative of the obtained integral must be equal to the given integrand.

Listed below are the integration formulas which find practical applications. These formulas can be verified by proving that the derivative of the resulting integral equals the given integrand.

Let u be a function of x.

1.	$\int u^n du = \frac{u^{n+1}}{n+1} + C$, $n \neq -1$
2.	$\int \frac{du}{u} = \ln\lvert u\rvert + C$
3.	$\int e^u du = e^u + C$, $e = 2.71828\ \ldots$
4.	$\int a^u = \frac{a^u}{\ln a} + C$, $a > 0, a \neq 1$
5.	$\int \sin u du = -\cos u + C$
6.	$\int \cos u du = \sin u + C$
7.	$\int \sec^2 u du = \tan u + C$
8.	$\int \csc^2 u du = -\cot u + C$

9.	$\int \sec u \tan u du = \sec u + C$
10.	$\int \csc u \cot u du = -\csc u + C$
11.	$\int \tan u du = \ln\lvert\sec u\rvert + C = -\ln\lvert\cos u\rvert + C$
12.	$\int \cot u du = \ln\lvert\sin u\rvert + C = -\ln\lvert\csc u\rvert + C$
13.	$\int \sec u du = \ln\lvert\sec u + \tan u\rvert + C$
14.	$\int \csc u du = \ln\lvert\csc u - \cot u\rvert + C$

3.3.1. General Power Formula

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad n \neq -1$$

Example 2. Evaluate the following indefinite integrals.

a) $\int (2x+7)(x^2+7x+3)^6 dx$

Observe that the derivative of the given function x^2+7x+3 appears on the given, hence, we can use the power formula where:

$n = 6$,

$u = x^2+7x+3$,

$\frac{du}{dx} = 2x+7$, or, differential $du = (2x+7)dx$

Likewise, take note that the needed du on the general power formula is complete.

Therefore, $\int (2x+7)(x^2+7x+3)^6 dx = \frac{(x^2+7x+3)^{6+1}}{6+1} + C = \frac{1}{7}(x^2+7x+3)^7 + C$

b). $\int \cos x(1+\sin x)^3 dx$

Analyzing whether the power formula works to integrate the given function, the $n = 3$

,

$u = 1+\sin x$,

$\frac{du}{dx} = \cos x$, or, differential $du = \cos x dx$

The needed du, observe, is again complete.

c) $\int e^{2x}\left(e^{2x}-4\right)^4 dx$

Analysis shows that the general power formula works to integrate the given function. Here, $n=4$,

$u=e^{2x}-4$

$\frac{du}{dx}=e^{2x}(2)=2e^{2x}$, or, differential $du=2e^{2x}dx$

Note that the needed du is not complete. There is a missing constant of 2. We need to introduce a constant factor of 2 in the integrand to complete the needed du. However, to neutralize its effect, it is necessary to introduce $\frac{1}{2}$ as well, the reciprocal of the missing constant, before the integral sign.

Then, we apply the general power formula. Hence,

$$\int e^{2x}\left(e^{2x}-4\right)^4 dx=\frac{1}{2}\int 2e^{2x}\left(e^{2x}-4\right)^4 dx$$

$$=\frac{1}{2}\left[\frac{\left(e^{2x}-4\right)^5}{5}\right]+C=\frac{1}{10}\left(e^{2x}-4\right)^5+C$$

d) $\int \sqrt{1-4x}\,dx$

The given radical can be transformed to equivalent power with fractional exponent. That is,

$$\int \sqrt{1-4x}\,dx=\int (1-4x)^{\frac{1}{2}}dx$$

Here, $n=\frac{1}{2}$, $u=1-4x$, $\frac{du}{dx}=-4$, or, differential $du=-4dx$.

Again, on this example, the needed du is incomplete with constant -4 missing. We need to introduce a constant factor of -4 in the integrand to complete the needed du. However, to neutralize its effect, it is necessary to introduce $\frac{1}{-4}=-\frac{1}{4}$ as well, the reciprocal of the missing constant, before the integral sign. Then, we apply the general power formula. Hence,

$$\int \sqrt{1-4x}\,dx=\int (1-4x)^{\frac{1}{2}}dx=-\frac{1}{4}\int -4(1-4x)^{\frac{1}{2}}dx$$

$$=-\frac{1}{4}\left[\frac{(1-4x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+C=-\frac{1}{4}\left[\frac{(1-4x)^{\frac{3}{2}}}{\frac{3}{2}}\right]+C$$

$$= -\frac{1}{4}\left[\frac{2}{3}\sqrt{(1-4x)^3}\right] + C = \frac{1}{6}(1-4x)\sqrt{1-4x} + C$$

e) $\int \frac{x^2}{(3-2x^3)^2}dx$

Before using the general power formula, using the definition of negative exponent, rewrite the given function in a way that the power will appear on the numerator. Hence,

$$\int \frac{x^2}{(3-2x^3)^2}dx = \int x^2(3-2x^3)^{-2}\,dx$$

On the power formula, $n = -2$, $u = 3-2x^3$, $\frac{du}{dx} = -6x^2$, or, differential $du = -6x^2dx$. Note that the needed du is incomplete with missing constant of -6. Hence, upon the introduction of its reciprocal, that is $-\frac{1}{6}$, before the integral sign, du is made complete. Therefore,

$$\int \frac{x^2}{(3-2x^3)^2}dx = \int x^2(3-2x^3)^{-2}\,dx = -\frac{1}{6}\int -6x^2(3-2x^3)^{-2}\,dx$$

$$= -\frac{1}{6}\left[\frac{(3-2x^3)^{-2+1}}{-2+1}\right] + C = -\frac{1}{6}\left[\frac{(3-2x^3)^{-1}}{-1}\right] + C$$

$$= \frac{1}{6(3-2x^3)} + C$$

f) $\int \frac{\sqrt[4]{1+\ln x^2}}{x}dx$

Using the power formula requires us to rewrite the radical as power having a fractional exponent.

$$\int \frac{\sqrt[4]{1+\ln x^2}}{x}dx = \int \frac{(1+\ln x^2)^{\frac{1}{4}}}{x}dx$$

Note that $n = \frac{1}{4}$, $u = 1+\ln x^2$, $\frac{du}{dx} = \frac{1}{x^2}(2x) = \frac{2}{x} = 2\left(\frac{1}{x}\right)$, or, differential $du = 2\left(\frac{dx}{x}\right)$.

Again, introduction of $\frac{1}{2}$ before the integral sign makes the needed du complete. Hence.

$$\int \frac{\sqrt[4]{1+\ln x^2}}{x}dx = \int \frac{(1+\ln x^2)^{\frac{1}{4}}}{x}dx = \frac{1}{2}\int (2)\frac{(1+\ln x^2)^{\frac{1}{4}}}{x}dx$$

$$= \frac{1}{2}\left[\frac{\left(1+\ln x^2\right)^{\frac{1}{4}+1}}{\frac{1}{4}+1}\right] + C = \frac{1}{2}\left[\frac{\left(1+\ln x^2\right)^{\frac{5}{4}}}{\frac{5}{4}}\right] + C$$

$$= \frac{1}{2}\left[\frac{4}{5}\sqrt[4]{\left(1+\ln x^2\right)^5}\right] + C = \frac{2}{5}\sqrt[4]{\left(1+\ln x^2\right)^5}$$

$$= \frac{2}{5}\left(1+\ln x^2\right)\sqrt[4]{1+\ln x^2} + C$$

3.3.2. The Logarithmic Function

The limitation of the general power formula $\int u^n du = \frac{u^{n+1}}{n+1} + C$ is when $n = 1$. This makes the right side of the equation indeterminate. This is where the logarithmic function comes in. Note that $\int u^{-1} du = \int \frac{du}{u}$, and we recall that $d(\ln u) = \frac{du}{u}$. Hence,

$$\boxed{\int \frac{du}{u} = \ln u + C}$$

The formula above involves numerator which is the derivative of the denominator. The denominator u represents any function involving any independent variable. The formula is meaningless when u is negative since the logarithm of negative numbers have not been defined. When negative numbers are involved, the formula should be considered in the form

$$\boxed{\int \frac{du}{u} = \ln|u| + C}$$

Therefore, the integral of a quotient whose numerator is the differential of the denominator is the logarithm of the denominator.

Example 3. Evaluate the following indefinite integrals.

a) $\int \frac{x^4}{2x^5 - 9} dx$

Using the formula $\int \frac{du}{u} = \ln|u| + C$, $u = 2x^5 - 9$, $\frac{du}{dx} = 10x^4$, or, $du = 10x^4 dx$, Hence, the needed du is incomplete, with constant 10 missing. There is a need to introduce $\frac{1}{10}$ before the integral sign.

$$\int \frac{x^4}{2x^5-9}dx = \frac{1}{10}\int \frac{10x^4}{2x^5-9}dx = \frac{1}{10}\ln\left|2x^5-9\right| + C$$

b) $\int \frac{\left(x^2-2\right)}{x^3-6x+4}dx$

The denominator $u = x^3 - 6x + 4$, $\frac{du}{dx} = 3x^2 - 6 = 3\left(x^2-2\right)$, or, $du = 3\left(x^2-2\right)dx$.

There is the missing constant of 3 on the needed du, hence, we introduce $\frac{1}{3}$ before the integral sign.

$$\int \frac{\left(x^2-2\right)}{x^3-6x+4}dx = \frac{1}{3}\int \frac{3\left(x^2-2\right)}{x^3-6x+4}dx = \frac{1}{3}\ln\left(x^3-6x+4\right) + C$$

c) $\int \frac{\cos 2x}{2+5\sin 2x}dx$

The denominator $u = 2 + 5\sin 2x$, $\frac{du}{dx} = 5\left(\cos 2x\right)\left(2\right) = 10\cos 2x$,or, $du = 10\cos 2x dx$. We need to introduce $\frac{1}{10}$ to complete the needed du.

Therefore, $\int \frac{\cos 2x}{2+5\sin 2x}dx = \frac{1}{10}\int \frac{10\cos 2x}{2+5\sin 2x}dx = \frac{1}{10}\ln\left|2+5\sin 2x\right| + C$

d) $\int \frac{\left(x^2+2\right)}{x+1}dx$

If the given function is an improper fraction, the first thing to do is to divide the numerator by the denominator, then, the formula $\int \frac{du}{u} = \ln\left|u\right| + C$ applies to integrate the remainder over the denominator.

$$\int \frac{\left(x^2+2\right)}{x+1}dx = \int \left(x - 1 + \frac{3}{x+1}\right)dx = \int x dx - \int dx + \int \frac{3}{x+1}dx$$

$$= \frac{x^2}{2} - x + 3\ln|x+1| + C = \frac{1}{2}x^2 - x + 3\ln|x+1| + C$$

e) $\int \frac{\sec x \tan x}{4 - 3\sec x} dx$

The denominator $u = 4 - 3\sec x$, $\frac{du}{dx} = -3\sec x \tan x$, or, $du = -3\sec x \tan x dx$. We introduce $\frac{1}{-3}$ before the integral sign to complete the needed du. Hence,

$$\int \frac{\sec x \tan x}{4 - 3\sec x} dx = -\frac{1}{3}\int \frac{-3\sec x \tan x}{4 - 3\sec x} dx = -\frac{1}{3}\ln|4 - 3\sec x| + C$$

f) $\int \frac{(\sin x + \tan x)}{\cos x} dx$

Transform the integrand into a form that is integrable. That s,

$$\int \frac{(\sin x + \tan x)}{\cos x} dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\tan x}{\cos x} dx$$

The first term is integrable using $\int \frac{du}{u} = \ln|u| + C$, where $u = \cos x$, $\frac{du}{dx} = -\sin x$, or, $du = -\sin x dx$. We need to introduce constant factor -1 before the integral sign to complete the needed du. Look how the second term is reduced to a form integrable.

$$\int \frac{(\sin x + \tan x)}{\cos x} dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\tan x}{\cos x} dx = -\ln|\cos x| + \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$$

$$= -\ln|\cos x| + \int \frac{\sin x}{\cos^2 x} dx = -\ln|\cos x| + \int \sin x (\cos x)^{-2} dx$$

Note that the second term is now integrable using the general power formula, where $n = -2$, $u = \cos x$, $\frac{du}{dx} = -\sin x$, or, $du = -\sin x dx$. Again, we introduce -1 before the integral sign to complete the needed du. Thus,

$$= -\ln|\cos x| + (-1)\int -\sin x (\cos x)^{-2} dx = -\ln|\cos x| - \frac{(\cos x)^{-2+1}}{-2+1} + C$$

$$= -\ln|\cos x| - \frac{(\cos x)^{-1}}{-1} + C = -\ln|\cos x| + \frac{1}{\cos x} + C = -\ln|\cos x| + \sec x + C$$

3.3.3. The Exponential Functions

$$\int e^u du = e^u + C \quad e = 2.71828 \ldots..$$

$$\int a^u = \frac{a^u}{\ln a} + C \quad a > 0, a \neq 1$$

Observe that in both formulas, u is an exponent. This makes the use of the formulas different on that of the general power formula wherein the exponent is a constant.

Example 4. Evaluate the following indefinite integrals.

a) $\int xe^{2x^2-1}dx$

On the formula $\int e^u du = e^u + C$, $u = 2x^2 - 1$, $\frac{du}{dx} = 4x$, or, $du = 4xdx$.

To complete the needed du, we introduce $\frac{1}{4}$ before the integral sign. Hence,

$$\int xe^{2x^2-1}dx = \frac{1}{4}\int (4)xe^{2x^2-1}dx = \frac{1}{4}e^{2x^2-1} + C$$

b) $\int (x-4)e^{x^2-8x+3}dx$

On the formula $\int e^u du = e^u + C$, $u = x^2 - 8x + 3$, $\frac{du}{dx} = 2x - 8 = 2(x-4)$, or,

$du = 2(x-4)dx$. Hence, to complete the needed du, we introduce $\frac{1}{2}$ before the integral sign. Thus,

$$\int (x-4)e^{x^2-8x+3}dx = \frac{1}{2}\int 2(x-4)e^{x^2-8x+3}dx = \frac{1}{2}e^{x^2-8x+3} + C$$

c) $\int \sec^2 3x\left(4^{\tan 3x}\right)dx$

This time, we use the formula $\int a^u = \frac{a^u}{\ln a} + C$, where $a = 4$, $u = \tan 3x$,

$\frac{du}{dx} = (\sec^2 3x)(3dx) = 3\sec^2 3xdx$. Observe that the needed du lacks constant 3.

Therefore, we introduce $\frac{1}{3}$ before the integral sign. Hence,

$$\int \sec^2 3x\left(4^{\tan 3x}\right)dx = \frac{1}{3}\int 3\sec^2 3x\left(4^{\tan 3x}\right)dx = \frac{1}{3}\cdot\frac{4^{\tan 3x}}{\ln 4} + C$$

This can further be simplified by using the law of logarithm $\log_b M^n = n\log_b M$. Thus, finally,

$$\int \sec^2 3x\left(4^{\tan 3x}\right)dx = \frac{1}{3}\cdot\frac{4^{\tan 3x}}{2\ln 2} + C = \frac{1}{6\ln 2}4^{\tan 3x} + C.$$

d) $\int \frac{xe^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}dx$

On the formula $\int e^u du = e^u + C$, $u = \sqrt{4+x^2}$,

$\frac{du}{dx} = \frac{1}{2\sqrt{4+x^2}}(2x) = \frac{x}{\sqrt{4+x^2}}$, or, $du = \frac{x}{\sqrt{4+x^2}}dx$. Therefore, the needed du is complete. Hence,

$$\int \frac{xe^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}dx = e^{\sqrt{4+x^2}} + C$$

e) $\int \frac{\left(e^x+1\right)^2}{e^x}dx$

Analysis shows that the formula $\int e^u du = e^u + C$ cannot be directly used. There is the need to reduce the given function to integrable form by expanding the square of a binomial on the numerator of the fraction, then, dividing the result by the denominator.

$$\int \frac{\left(e^x+1\right)^2}{e^x}dx = \int \frac{e^{2x}+2e^x+1}{e^x}dx = \int e^x dx + \int 2dx + \int \frac{dx}{e^x}$$

The definition of negative exponent is then applied on the last term.

$$\int \frac{\left(e^x+1\right)^2}{e^x}dx = \int e^x dx + \int 2dx + \int e^{-x}dx = e^x + 2x + (-1)\int -e^{-x}dx$$

$$\int \frac{\left(e^x+1\right)^2}{e^x}dx = e^x + 2x - e^{-x} + C = e^x + 2x - \frac{1}{e^x} + C$$

f) $\int e^{2x}\sqrt{1+e^{2x}}dx$

Take note that the formula $\int e^u du = e^u + C$ will not work to evaluate the given integral. The presence of the exponential function e^{2x} does not necessary mean that formula can do the antidifferentiation task. However, a closer look tells that the genera; power formula could be an effective tool to do the integration process with the $n=\frac{1}{2}$, $u=1+e^{2x}$, $\frac{du}{dx}=e^{2x}(2dx)=2e^{2x}dx$, or, $du=2e^{2x}dx$. Thus, we need to introduce $\frac{1}{2}$ before the integral sign.

$$\int e^{2x}\sqrt{1+e^{2x}}dx=\frac{1}{2}\int 2e^{2x}\sqrt{1+e^{2x}}dx=\frac{1}{2}\left[\frac{(1+e^{2x})^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+C$$

$$=\frac{1}{2}\left[\frac{(1+e^{2x})^{\frac{3}{2}}}{\frac{3}{2}}\right]+C=\frac{1}{2}\left[\frac{2}{3}(1+e^{2x})^{\frac{3}{2}}\right]+C$$

$$=\frac{1}{3}\sqrt{(1+e^{2x})^{3}}+C=\frac{1}{3}(1+e^{2x})\sqrt{1+e^{2x}}+C$$

g) $\int 3^{4x}e^{4x}dx$

We first reduce to integrable form the given function using the rule of exponents $(ab)^m=a^m b^m$, then use the formula $\int a^u=\frac{a^u}{\ln a}+C$ where $a=3e$, $u=4x$, $\frac{du}{dx}=4$, or, $du=4dx$. Note that we need to introduce $\frac{1}{4}$ before the integral sign.

$$\int 3^{4x}e^{4x}dx=\int (3e)^{4x}dx=\frac{1}{4}\int 4(3e)^{4x}dx=\frac{1}{4}\cdot\frac{(3e)^{4x}}{\ln 3e}+C$$

However, $\ln 3e=\ln 3+\ln e=\ln 3+1$ based on the laws of logarithms $\ln e=1$ and $\log_b MN=\log_b M+\log_b N$. Therefore,

$$\int 3^{4x}e^{4x}dx=\frac{(3e)^{4x}}{4(1+\ln 3)}+C=\frac{3^{4x}e^{4x}}{4(1+\ln 3)}+C$$

3.3.4. The Trigonometric Functions

1. $\int \sin u du=-\cos u+C$
2. $\int \cos u du=\sin u+C$

3.	$\int \sec^2 u du = \tan u + C$
4.	$\int \csc^2 u du = -\cot u + C$
5.	$\int \sec u \tan u du = \sec u + C$
6.	$\int \csc u \cot u du = -\csc u + C$
7.	$\int \tan u du = \ln\lvert\sec u\rvert + C = -\ln\lvert\cos u\rvert + C$
8.	$\int \cot u du = \ln\lvert\sin u\rvert + C = -\ln\lvert\csc u\rvert + C$
9.	$\int \sec u du = \ln\lvert\sec u + \tan u\rvert + C$
10.	$\int \csc u du = \ln\lvert\csc u - \cot u\rvert + C$

Observe that the above antidifferentiation formulas are derived directly from their corresponding differentiation formulas. To show derivation of Formula 7 $\int \tan u du = \ln\lvert\sec u\rvert + C = -\ln\lvert\cos u\rvert + C$, we express $\tan u = \frac{\sec u \tan u}{\sec u}$ and then, apply the formula $\int \frac{du}{u} = \ln\lvert u\rvert + C$. Hence, $\int \tan u du = \int \frac{\sec u \tan u}{\sec u} du = \ln \sec u + C$.

Similarly, derivation can be done by expressing, this time, $\tan u = \frac{\sin u}{\cos u}$. Thus, $\int \tan u du = \int \frac{\sin u}{\cos u} du = -\ln \cos u + C$.

Furthermore, Formula 9 $\int \sec u du = \ln\lvert\sec u + \tan u\rvert + C$ can be derived using Formula $\int \frac{du}{u} = \ln\lvert u\rvert + C$. Let the $u = \sec u + \tan u$, $\frac{du}{dx} = \sec u \tan u + \sec^2 u$, or, $du == \sec u(\sec u + \tan u)dx$. Other formulas can be derived in the same way.

Example 5. Evaluate the following indefinite integrals.

a) $\int \sin \frac{1}{4} x dx$

On the formula $\int \sin u du = -\cos u + C$, the $u = \frac{1}{4}x$, $\frac{du}{dx} = \frac{1}{4}$, or, $du = \frac{1}{4}dx$. We need to introduce 4 and the, apply the above formula. Hence,

$$\int \sin \frac{1}{4} x dx = 4\int \frac{1}{4} \sin \frac{1}{4} x dx = 4\left(-\cos \frac{1}{4} x\right) + C = -4\cos \frac{1}{4} x + C$$

b) $\int x\tan\left(4-x^2\right)dx$

We use formula $\int \tan u du = \ln\left|\sec u\right| + C = -\ln\left|\cos u\right| + C$ where $u = 4 - x^2$,

$\frac{du}{dx} = -2x$, or, $du = -2xdx$. There is the need to introduce $-\frac{1}{2}$ before the integral sign. Therefore,

$$\int x\tan\left(4-x^2\right)dx = -\frac{1}{2}\int -2x\tan\left(4-x^2\right)dx = -\frac{1}{2}\ln\left|\sec\left(4-x^2\right)\right| = \frac{1}{2}\ln\left|\cos\left(4-x^2\right)\right|.$$

c) $\int (2-\sec x)^2 dx$

This is not even integrable by the general power formula because du is incomplete not only by a constant but with $\sec x\tan x$. We expand the numerator which is square of a binomial using the special product $(a-b)^2 = a^2 - 2ab + b^2$. Hence,

$$\int (2-\sec x)^2 dx = \int \left(4 - 4\sec x + \sec^2 x\right)dx = \int 4dx - 4\int \sec xdx + \int \sec^2 xdx$$

Observe that Formulas 3 and 9 best apply to perform the integration work.

$$\int (2-\sec x)^2 dx = 4x - 4\ln\left|\sec x + \tan x\right| + \tan x + C$$

d) $\int \left(\sin^4 x - \cos^4 x\right)dx$

The expression $\left(\sin^4 x - \cos^4 x\right) = \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)$ is factored as a difference of two squares. Moreover, $\sin^2 x + \cos^2 x = 1$ based on Pythagorean identity. The factor $\left(\sin^2 x - \cos^2 x\right)$ can be simplified to $-\left(\cos^2 x - \sin^2 x\right)$ which is equal to $-\cos 2x$ with reference to double an angle identity, Antidifferentiation is done using Formula $\int \cos udu = \sin u + C$, where $u = 2x$, $\frac{du}{dx} = 2$, or, $du = 2dx$. Therefore, we need to introduce $\frac{1}{2}$ before the integral sign.

$$\int \left(\sin^4 x - \cos^4 x\right)dx = \int -\cos 2x = \frac{1}{2}\int 2(-\cos 2x)dx = -\frac{1}{2}\sin 2x + C$$

e) $\int \frac{(2\cos x + \sin x)^2}{\sin x}dx$

We got $4\cos^2 x + 4\sin x\cos x + \sin^2 x$ after expanding the square of binomial on the numerator. Then, based on the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$, we come up to $\cos^2 x = 1 - \sin^2 x$. Hence,

$$\int \frac{(2\cos x+\sin x)^2}{\sin x}dx = \int \frac{\left(4\cos^2 x + 4\sin x\cos x + \sin^2 x\right)}{\sin x}dx$$
$$= \int \frac{\left[4\left(1-\sin^2 x\right)+4\sin x\cos x+\sin^2 x\right]}{\sin x}dx = \int \frac{4-3\sin^2 x+4\sin x\cos x}{\sin x}dx$$
$$= \int \frac{4}{\sin x}dx - 3\int \frac{\sin^2 x}{\sin x}dx + 4\int \frac{\sin x\cos x}{\sin x}dx$$
$$= \int 4\csc x dx - 3\int \sin x dx + 4\int \cos x dx$$
$$= 4\ln\left|\csc x - \cot x\right| - 3(-\cos x) + 4\sin x + C$$
$$= 4\ln\left|\csc x - \cot x\right| + 3\cos x + 4\sin x + C$$

f) $\int \frac{\sin^2 2x}{1-\cos 2x}dx$

To reduce the given function to integrable form, we use Pythagorean Identity $\sin^2 2x = 1-\cos^2 2x$ which then is factored as a difference of two squares.

$$\int \frac{\sin^2 2x}{1-\cos 2x}dx = \int \frac{1-\cos^2 2x}{1-\cos 2x}dx = \int \frac{(1+\cos 2x)(1-\cos 2x)}{1-\cos 2x}dx$$
$$= \int (1+\cos 2x)dx = \int dx + \int \cos 2x dx$$
$$= x + \frac{1}{2}\int 2\cos 2x dx = x + \frac{1}{2}\sin 2x + C$$

3.4. Antidifferentiation Using Substitution Rule

The integration tasks done previously can be made a little easier by representing a part of the given integrand by a new variable and eliminating the original variable on the given function including its differential. This method of antidifferentiation or integration is what we call the Substitution Rule.

Example 6. Evaluate the following indefinite integrals using substitution rule.

a) $\int 6x\left(3x^2-1\right)^4 dx$

Part of the given integrand that can be replaced is $\left(3x^2-1\right)$.

Let: $u = 3x^2-1$, $\frac{du}{dx} = 6x$ .or, $6xdx = du$. Therefore,

$$\int 6x\left(3x^2-1\right)^4 dx = \int \left(3x^2-1\right)^4(6xdx) = \int u^4 du$$
$$= \frac{u^{4+1}}{4+1} + C = \frac{1}{5}u^5 + C = \frac{1}{5}\left(3x^2-1\right)^5 + C$$

b) $\int e^{4x}\left(e^{4x}-1\right)^3 dx$

Let $u=e^{4x}-1$, $\frac{du}{dx}=4e^{4x}$, or, $e^{4x}dx=\frac{du}{4}$. Substitution yields

$$\int e^{4x}\left(e^{4x}-1\right)^3 dx=\int\left(e^{4x}-1\right)^3\left(e^{4x}dx\right)=\int u^3\left(\frac{du}{4}\right)=\frac{1}{4}\int u^3 du=\frac{1}{4}\left(\frac{u^4}{4}\right)+C$$

$$=\frac{1}{16}u^4+C=\frac{1}{16}\left(e^{4x}-1\right)^4+C$$

c) $\int x^4\sin x^5 dx$

Let $u=x^5$, $\frac{du}{dx}=5x^4$, or, $x^4dx=\frac{du}{5}$. Substitution results to

$$\int x^4\sin x^5 dx=\int\left(\sin x^5\right)\left(x^4dx\right)=\int(\sin u)\left(\frac{du}{5}\right)=\frac{1}{5}\int\sin u du=\frac{1}{5}(-\cos u)+C$$

$$=-\frac{1}{5}\cos u+C=-\frac{1}{5}\cos x^5+C$$

d) $\int\frac{dx}{x\sqrt{\ln x-4}}$

Let $u=\ln x-4$, $\frac{du}{dx}=\frac{1}{x}$, $\frac{dx}{x}=du$. Hence,

$$\int\frac{1}{\sqrt{\ln x-4}}\left(\frac{dx}{x}\right)=\int\frac{1}{\sqrt{u}}du=\int\frac{1}{u^{\frac{1}{2}}}du=\int u^{-\frac{1}{2}}du$$

$$=\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C=2u^{\frac{1}{2}}$$

$$=2\sqrt{u}+C=2\sqrt{\ln x-4}+C$$

e) $\int\csc x\cot x(1-\csc x)^5 dx$

Let $u=1-\csc x$, $\frac{du}{dx}=-(-\csc x\cot x)=\csc x\cot x$, or, $du=\csc x\cot x dx$. Therefore,

$$\int(1-\csc x)^5(\csc x\cot x dx)=\int u^5 du=\frac{u^6}{6}+C=\frac{1}{6}(1-\csc x)^6+C$$

f) $\int \frac{x^5 e^{\sqrt{x^6-1}}}{\sqrt{x^6-1}}dx$

Let $u=\sqrt{x^6-1}$, $\frac{du}{dx}=\frac{1}{2\sqrt{x^6-1}}\left(6x^5\right)=\frac{3x^5}{\sqrt{x^6-1}}$, or, $\frac{du}{3}=\frac{x^5dx}{\sqrt{x^6-1}}$. Therefore,

$$\int \frac{x^5 e^{\sqrt{x^6-1}}}{\sqrt{x^6-1}}dx=\int e^{\sqrt{x^6-1}}\left(\frac{x^5dx}{\sqrt{x^6-1}}\right)=\int e^u\left(\frac{du}{3}\right)$$

$$=\frac{1}{3}\int e^u du=\frac{1}{3}e^u+C=\frac{1}{3}e^{\sqrt{x^6-1}}+C$$

3.5. Table of Integrals Involving Inverse Trigonometric Functions and Logarithms

1	$\int \frac{du}{\sqrt{a^2-u^2}}=Sin^{-1}\frac{u}{a}+C$
2	$\int \frac{du}{a^2+u^2}=\frac{1}{a}Tan^{-1}\frac{u}{a}+C$
3	$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}Sec^{-1}\frac{u}{a}+C$
4	$\int \frac{du}{\sqrt{u^2\pm a^2}}du=\ln\left(u+\sqrt{u^2\pm a^2}\right)+C$
5	$\int \frac{du}{a^2-u^2}=\frac{1}{2a}\ln\frac{a+u}{a-u}+C$, $u^2<a^2$
6	$\int \frac{du}{u^2-a^2}=\frac{1}{2a}\ln\frac{u-a}{u+a}+C$, $u^2>a^2$
7	$\int \sqrt{a^2-u^2}du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}Sin^{-1}\frac{u}{a}+C$
8	$\int \sqrt{u^2\pm a^2}du=\frac{u}{2}\sqrt{u^2\pm a^2}+\frac{a^2}{2}\ln\left(u+\sqrt{u^2\pm a^2}\right)+C$

Example 7. Evaluate the following indefinite integrals using the table of integrals.

a) $\int \frac{x}{\sqrt{4-x^4}}dx$

On the Formula $\int \frac{du}{\sqrt{a^2-u^2}}=Sin^{-1}\frac{u}{a}+C$, $a^2=4$, **or,** $a=2$, $u^2=x^4$, **or,** $u=x^2$

$\frac{du}{dx} = 2x$, or, $du = 2xdx$. The needed du is incomplete, with 2 as missing constant.

Hence, we introduce $\frac{1}{2}$ before the integral sign.

$$\int \frac{x}{\sqrt{4-x^4}} dx = \frac{1}{2}\int \frac{2x}{\sqrt{4-x^4}} dx = \frac{1}{2} Sin^{-1} \frac{x^2}{2} + C$$

b) $\int \frac{1}{1+16x^2} dx$

On the Formula $\int \frac{du}{a^2+u^2} = \frac{1}{a} Tan^{-1} \frac{u}{a} + C$, $a^2 = 1$, or, $a = 1$, $u^2 = 16x^2$, or, $u = 4x$, $\frac{du}{dx} = 4$, or, $du = 4dx$. Therefore, there is a need to introduce $\frac{1}{4}$ before the integral sign. Hence,

$$\int \frac{1}{1+16x^2} dx = \frac{1}{4}\int \frac{4dx}{1+16x^2} = \frac{1}{4}\left(\frac{1}{1}\right) Tan^{-1} \frac{4x}{1} + C = \frac{1}{4} Tan^{-1} 4x + C$$

c) $\int \frac{x^2}{9-x^6} dx$

On the Formula $\int \frac{du}{a^2-u^2} = \frac{1}{2a} \ln \frac{a+u}{a-u} + C$, $a^2 = 9$, or, $a = 3$, $u^2 = x^6$, or, $u = x^3$, $\frac{du}{dx} = 3x^2$, or, $du = 3x^2dx$. It is necessary to introduce $\frac{1}{3}$ before the integral sign.

$$\int \frac{x^2}{9-x^6} dx = \frac{1}{2(3)} \ln \frac{3+x^3}{3-x^3} + C = \frac{1}{6} \ln \frac{3+x^3}{3-x^3} + C$$

d) $\int \frac{\cos 2x}{\sin^2 2x - 4} dx$

On the Formula $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \frac{u-a}{u+a} + C$, $a^2 = 4$, or, $a = 2$, $u^2 = \sin^2 2x$, or, $u = \sin 2x$, $\frac{du}{dx} = \cos 2x(2) = 2\cos 2x$, or, $du = 2\cos 2xdx$. Thus,

$$\int \frac{\cos 2x}{\sin^2 2x - 4} dx = \frac{1}{2}\int \frac{2\cos 2x dx}{\sin^2 2x - 4}$$

$$= \frac{1}{2}\left[\frac{1}{2(2)} \ln \frac{\sin 2x - 2}{\sin 2x + 2}\right] + C$$

$$= \frac{1}{8} \ln \frac{\sin 2x - 2}{\sin 2x + 2} + C$$

e) $e^{2x}\sqrt{25 - e^{4x}}dx$

On the Formula $\int \sqrt{a^2 - u^2}du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} Sin^{-1}\frac{u}{a} + C$, $a^2 = 25$, **or,** $a = 5$, $u^2 = e^{4x}$, **or,** $u = e^{2x}$, $\frac{du}{dx} = 2e^{2x}$, **or,** $du = 2e^{2x}dx$. There is a need to introduce $\frac{1}{2}$ before the integral sign. Thus,

$$\int e^{2x}\sqrt{25 - e^{4x}}dx = \frac{1}{2}\int 2e^{2x}\sqrt{25 - e^{4x}}dx = \frac{1}{2}\left[\frac{e^{2x}}{2}\sqrt{25 - e^{4x}} + \frac{25}{2} Sin^{-1}\frac{e^{2x}}{5}\right] + C$$

$$= \frac{1}{4}\left[e^{2x}\sqrt{25 - e^{4x}} + 25 Sin^{-1}\frac{e^{2x}}{5}\right] + C$$

f) $\int \sqrt{4 - (1 + x)^2}dx$

On the Formula $\int \sqrt{u^2 \pm a^2}du = \frac{u}{2}\sqrt{u^2 \pm a^2} + \frac{a^2}{2}\ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$, $a^2 = 4$, **or,** $a = 2$, $u^2 = (1 + x)^2$, **or,** $u = 1 + x$, $du = dx$. Therefore,

$$\int \sqrt{4 - (1 + x)^2}dx = \frac{1 + x}{2}\sqrt{4 - (1 + x)^2} + \frac{4}{2}\ln\left(1 + x + \sqrt{4 - (1 + x)^2}\right) + C$$

$$= \frac{1 + x}{2}\sqrt{4 - (1 + x)^2 + 2\ln\left(1 + x + \sqrt{4 - (1 + x)^2}\right) + C}$$

3.6. Differential Equations

A differential equation is an equation involving an unknown function and its derivatives. The order of the differential equation is the order of the highest derivative of the unknown function involved in the equation. In this section, we will talk about an ordinary differential equation (ODE) which is defined as a differential equation with only one dependent and one independent variable. Listed below are examples of ODE of order one.

a) $\frac{dy}{dx} = \frac{2y}{3x}$ b) $\frac{dy}{dx} = \cos x$ c) $\frac{dy}{dx} = \frac{1+y^2}{2xy}$

The solution of a differential equation is an expression of the dependent variable in terms of the independent variable which satisfies the differential equation. The general idea is that, instead of solving equations to find unknown numbers, here, we will solve equations to find unknown *functions*. Hence, to solve a differential equation means finding its solution which is of two kinds:

1. General solution which contains a number of arbitrary constants equal to the order of the differential equation.
2. Particular solution which is obtained from the general solution by specifying the value/s of the arbitrary constant/s.

The differential equation of the form $\frac{dy}{dx} = f(x, y)$ is called separable equation if $f(x, y) = h(x)g(y)$, hence, $\frac{dy}{dx} = h(x)g(y)$. This discussion will show how to solve a given separable differential equation using antidifferentiation by performing the following steps.

a) Bring the given differentia equation to the form $\frac{dy}{dx} = h(x)g(y)$.

b) Rewrite it to the form $\frac{dy}{g(y)} = h(x)dx$. Observe that the coefficient of dy is in terms of y while the coefficient of dx in terms of x alone.

c) Integrate or anti-differentiate the equation in Step b) to obtain the solution of the given separable equation.

d) If condition is given, evaluate the constant of integration on the general solution to find the particular solution of the given differential equation.

Example 8. Solve the following separable differential equations.

a) $\frac{dy}{dx} = 2x - 5$, when $x = 0, y = 2$

First, we bring the given equation to the form $\frac{dy}{g(y)} = h(x)dx$. That is.

$$dy = (2x-5)dx$$

Then, we integrate both sides of the above equation.

$$\int dy = \int (2x-5)dx$$

$$\int dy = \int 2x dx - 5\int dx$$

Apply the appropriate integration formula.

$$y = 2\left(\frac{x^2}{2}\right) - 5x + C$$

$$y = x^2 - 5x + C$$

Since initial condition is given, that is, when $x = 0, y = 2$, we evaluate C, the constant of integration (arbitrary constant). Hence,

$$2 = 0 + 0 + C$$
$$C = 2$$

Therefore, the particular solution of the given differential equation is $y = x^2 - 5x + 2$.

b) $e^y - x^2 \frac{dy}{dx} = 0$, when $x = 2, y = 0$

Reduce the given differential equation to the form $\frac{dy}{g(y)} = h(x)dx$ by separating the variables.

$$x^2 \frac{dy}{dx} = e^y$$

$$x^2 dy = e^y dx$$

$$\frac{dy}{e^y} = \frac{dx}{x^2}$$

Integrate both sides of the above equation.

$$\int \frac{dy}{e^y} = \int \frac{dx}{x^2}$$

$$\int e^{-y} dy = \int x^{-2} dx$$

Use the appropriate integration formula.

$$(-1)e^{-y} = \left(\frac{x^{-1}}{-1}\right) + C$$

$$-\frac{1}{e^y} = -\frac{1}{x} + C$$

Since condition is given, that is, when $x = 2, y = 0$, we evaluate C, the constant of integration (arbitrary constant). Hence,

$$-\frac{1}{e^0} = -\frac{1}{2} + C$$

$$C = \frac{1}{2} - 1 = -\frac{1}{2} \quad \text{(Note that } e^0 = 1\text{)}$$

Therefore, the particular solution of the given differential equation is:

$$-\frac{1}{e^y}=-\frac{1}{x}-\frac{1}{2}$$

Simplify the above particular solution. Multiply both sides of equation by $-2xe^y$.

$$2x=2e^y+xe^y$$
$$2x=e^y(2+x)$$

c) $\frac{dy}{dx}=-\frac{y}{x}$, when $x=1, y=1$

$$xdy=-ydx$$
$$\frac{dy}{y}=-\frac{dx}{x}$$
$$\int\frac{dy}{y}=-\int\frac{dx}{x}$$
$$\ln y=-\ln x+C$$

Apply the condition $x=1, y=1$ to evaluate C.

$$\ln 1=-\ln 1+C$$

But, $\ln 1=0$. $\qquad C=0$

Therefore, the particular solution is:

$$\ln y=-\ln x$$

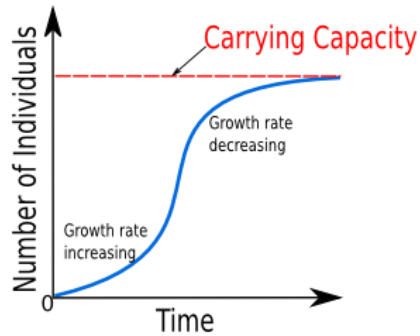
Simplify.

$$\ln y+\ln x=0$$
$$\ln xy=0$$
$$e^{\ln xy}=e^0$$
$$xy=1$$

3.7. Exponential Growth and Decay, Bounded Growth and Logistic Growth

3.7.1. Exponential Growth and Decay

Population growth rate is the measure of how the size of a population changes over time. There will be an increase in the number of individuals in a population over time as long as there are enough resources available. However, as these resources begin to run out, population growth will start to slow down. When the growth rate of a population decreases as the number of individuals increases, this is called *logistic population growth*. Moreover, in logistic growth, a population will continue to grow until it reaches *carrying* *capacity*, the maximum number of individuals the environment can support. Below is the graph of logistic population growth.



Bounded growth is when a growth rate is constantly decreasing. The result is that growth forever approaches a fixed value. Logistic growth is a bounded growth.

In exponential growth, the growth rate of a population increases over time, in proportion to the size of the population. Exponential growth may happen for a while, if there are few individuals and many resources. But when the number of individuals gets large enough, resources start to get used up, slowing the growth rate.

Solving exponential growth problems leads to a separable differential equation.

Let $N(t)$ be the population present at time t. This amount is either increasing (growing) or decreasing (decaying).

$\frac{dN}{dt}$ be the time-rate of change of the population.

N_0 be the amount of the substance when time $t=0$

Let us assume that $\frac{dN}{dt}$ is proportional to the amount N present. Hence,

$$\frac{dN}{dt} = kN$$

Where k is the proportionality constant. If the constant k is positive, it has exponential growth and if k is negative, then, it has exponential decay.

Separating the variables and integrating,

$$\int \frac{dN}{N} = \int k\,dt$$

$$\ln N = kt + C$$

Use the condition that when $t=0, N=N_0$. This is to evaluate constant C on the general solution.

$$\ln N_0 = k(0) + C \qquad C = \ln N_0$$

Therefore, the particular solution is

$$\ln N = kt + \ln N_0$$

Simplify. $\ln N - \ln N_0 = kt$

Use $\log_b M - \log_b N = \log_b \frac{M}{N}$: $\ln \frac{N}{N_0} = kt$

$$e^{\ln \frac{N}{N_0}} = e^{kt}$$

Use $e^{\ln \frac{N}{N_0}} = \frac{N}{N_0}$. $\frac{N}{N_0} = e^{kt}$

$$\boxed{N = N_0 e^{kt}}$$

The equation in box is a useful one when we want to know the population size at a particular time.

Example 9. The number of bacteria in a liquid culture is observed to grow exponentially at a rate proportional to the number of cells present. At the beginning of the experiment, there are 10,000 cells and after three hours there are 500,000. How many will there be after one day of growth if this unlimited growth continues?

Solution:

This is an exponential growth problem. The given conditions are when $t = 0, N = 10{,}000$ and $t = 3$ hours, $N = 500{,}000$. We have to find N when $t = 1$ day or 24 hours.

We start using the boxed-equation: $N = N_0 e^{kt}$.

Substitute $N_0 = 10{,}000$: $N = 10{,}000\, e^{kt}$ ------- Equation A

Find now value of proportionality constant k using the condition that when $t = 3$ hours, $N = 500{,}000$. $500{,}000 = 10{,}000\, e^{k(3)}$

$$50 = e^{3k}$$

Take the ln of both sides of equaiton: $\ln 50 = \ln e^{3k}$

$$\ln 50 = 3k$$

$$k = \frac{1}{3} \ln 50 = 1.3$$ ------ Equation B

Substitute Equation B into Equation A. $N = 10{,}000\, e^{1.3t}$

Find N when $t = 24$ hours. $N = 10{,}000\, e^{1.3(24)}$

$$N = 3.55 x 10^{17}$$

Therefore, the number of bacteria in the liquid culture after one day is $3.55 x 10^{17}$.

Example 10. A colony of bacteria is growing exponentially. If there are $1x10^4$ bacteria at the end of 3 hours and $4x10^4$ at the end of 5 hours, how many were there at the beginning?

Solution:

Use the exponential growth equation. $N = N_0 e^{kt}$

When $t = 5$, $N = 4x10^4$. $4x10^4 = N_0 e^{k(5)}$

$4x10^4 = N_0 e^{5k}$ ------ Equation A

When $t = 3$, $N = 1x10^4$. $1x10^4 = N_0 e^{3k}$

$N_0 = \frac{1x10^4}{e^{3k}}$ -------- Equation B

Substitute Equation B into Equation A. $4x10^4 = \left(\frac{1x10^4}{e^{3k}}\right)\left(e^{5k}\right)$

$\frac{4x10^4}{1x10^4} = e^{5k-3k}$

$4 = e^{2k}$

Take ln of both sides of the equation. $\ln 4 = \ln e^{2k}$

$\ln 4 = 2k$

$k = \frac{1}{2}\ln 4 = 0.69$

Substitute into Equation B. $N_0 = \frac{1x10^4}{e^{3(0.69)}} = \frac{1x10^4}{e^{2.07}} = \frac{1x10^4}{7.92} = 1261$

Therefore, there are 1261 bacteria initially in the culture.

3.7.2. Logistic and Bounded Growth

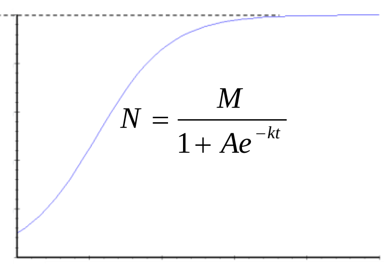
We have used the exponential growth equation $N = N_0 e^{kt}$ to represent population growth. The exponential growth equation occurs when the rate of growth is proportional to the amount present. If we use N to represent the population, the differential equation becomes: $\frac{dN}{dt} = kN$, the constant k is called the relative growth rate. The population growth model becomes, $N = N_0 e^{kt}$. However, real-life populations do not increase forever. There is some limiting factor such as food, living space or waste disposal. There is a maximum population, or carrying capacity, M . A more realistic application in differential equation is the logistic growth model where growth rate is proportional to both the amount present N and the carrying capacity that remains: $1 - \frac{N}{M}$.

The equation then becomes:

Logistic Differential Equation

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{M}\right)$$

A population that grows according to this model does not grow without bound, wherein $0 < N < M$, the population increases since $\frac{dN}{dt} > 0$. On the other hand when $N > M$, the population decreases because $\frac{dN}{dt} < 0$. Showing the graph of function N that satisfies a logistic differential equation.



We can solve this differential equation $\frac{dN}{dt} = kN\left(1 - \frac{N}{M}\right)$ which is separable equation.

$$\int \frac{dN}{N\left(1 - \frac{N}{M}\right)} = \int k\,dt$$

In order to evaluate the left hand side, we write:

$\frac{1}{N\left(1 - \frac{N}{M}\right)} = \frac{M}{N(M - N)} = \frac{1}{N} + \frac{1}{M - N}$, Hence, $\int \frac{dN}{N} + \int \frac{dN}{M - N} = \int k\,dt,$

$\ln|N| - \ln|M - N| = kt + C$

$\ln|M - N| - \ln|N| = -kt - C$

$\ln\left|\frac{M - N}{N}\right| = -kt - C$

$\left|\frac{M - N}{N}\right| = e^{-kt - C}$

$\frac{M - N}{N} = e^{-kt - C}$, Apply the property $e^{x+y} = e^x \bullet e^y$

$\frac{M-N}{N}=e^{-C}\bullet e^{-kt}$, let $A=\pm e^{-C}$, $\frac{M-N}{N}=Ae^{-kt}$

From here we get:

Logistic Growth Model

$$N=\frac{M}{1+Ae^{-kt}}$$

Example 11: Ten grizzly bears were introduced to a national park 10 years ago. There are 25 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 50? 75?

Solution: Use the logistic growth model $N=\frac{M}{1+Ae^{-kt}}$

$M=100$, $N_0=10$, $N_{10}=25$

At time zero, the population is 10 grizzly bears

$$10=\frac{100}{1+Ae^{0}}$$

$$10=\frac{100}{1+A}$$

$$10(1+A)=100$$

$$10+10A=100$$

$$10A=90$$

$$A=9$$

Substitute the value of A to the logistic growth model,

$N=\frac{100}{1+9e^{-kt}}$ ....Equation 1

After 10 years, the population N is 25.

$$23=\frac{100}{1+9e^{-kt}}$$

$$1+9e^{-10k}=\frac{100}{25}$$

$$1+9e^{-10k}=4$$

$$9e^{-10k}=3$$

$$e^{-10k}=\frac{3}{9},\quad e^{-10k}=\frac{1}{3},$$

$e^{-10k} = 0.33$, $e^{-10k} = e^{\ln 0.33}$

Applying the law of exponent: $e^{x} = e^{y} \rightarrow x = y$

$-10k = \ln 0.33$

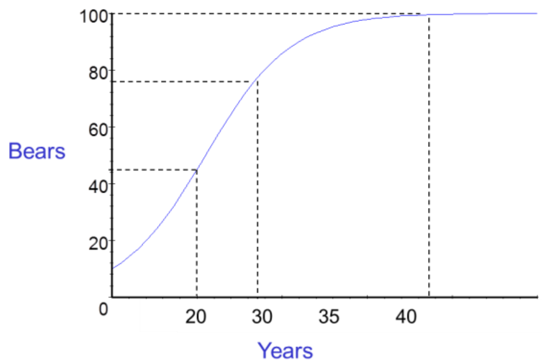
$-10k = -1.11$

$k = 0.11$

Substitute the value of k to equation 1, we get:

$$N = \frac{100}{1+9e^{-0.11t}}$$

We can graph this equation and use “trace” to find the solutions

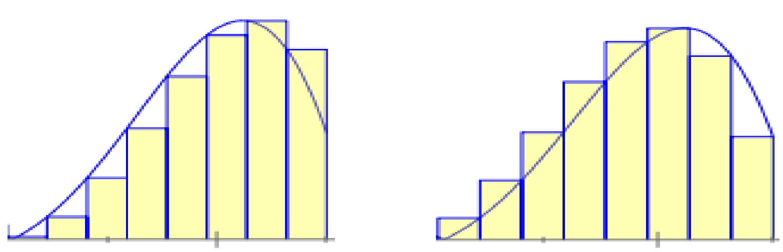


Therefore, 20 years, the bear population reach 50, while 29 years, the bear will reach 75.

3.8. Riemann Sum

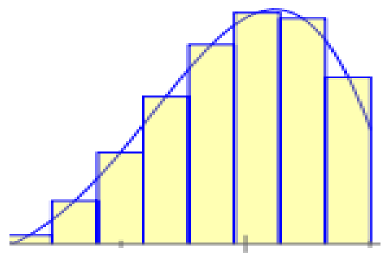
One of the various approximations that may be used to find the area under a curve is Riemann sum that fits one or more rectangles beneath a curve and takes the total area of those rectangles as the estimated area beneath the curve. If more than a single rectangle is used, it is often most desirable to have the rectangles be the same width so only their heights vary. This makes for quicker calculations especially when the number of rectangles is large.

To make a Riemann sum, we must choose how we're going to make our rectangles. One possible choice is to make our rectangles touch the curve with their top-left corners. This is called a *left Riemann sum*. Another choice is to make our rectangles touch the curve with their top-right corners. This is a *right Riemann sum.*



Observe on the above examples, the "left" approximation will be too small (the sum of the rectangle areas is less than the area below the curve), while the "right" one will be too large.

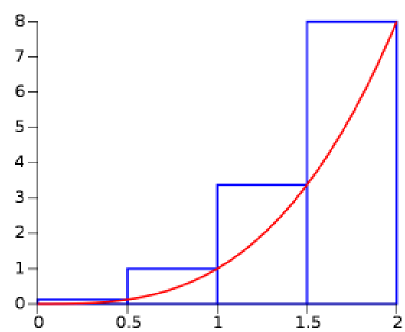
Another, and better option is to place the rectangles so the curve passes through the *mid-points* of each rectangle, as follows:



Example 12. Find the area under the curve from $x = 0$ to $x = 2$ for the function $y = x_3$ using the right Riemann sum.

Solution: We form rectangles of equal width between the start and endpoint of the area we need. Then, we make the rectangles touch the curve with their top-right corners. If we want to draw four rectangles of common width, the constant width of the rectangles equals $\left(\frac{2-0}{4}\right) = 0.5$,

.



Draw the rectangles using points furthest to the right. Place your pen on the endpoint (the first endpoint to the right is 0.5), draw up to the curve and then, draw left to the y-axis to form a rectangle. Then, calculate the area of each rectangle by multiplying the height by the constant width.

Interval	$0-0.5$	$0.5-1$	$1-1.5$	$1.5-2$
Height	$(0.5)^3 = 0.125$	$(1)^3 = 1$	$(1.5)^3 = 3.375$	$(2)^3 = 8$
Width	0.5	0.5	0.5	0.5
Height x Width	0.0625	0.5	1.6875	4
Area	$0.0625 + 0.5 + 1.6875 + 4 = 6.25$			

3.9.RIEMANN SUM FORMULA for the Definite Integral

Definite integrals represent the exact area under a given curve, and Riemann sums are used to approximate those areas. However, if we take Riemann sums with infinite rectangles of infinitely small width, we get the exact area, i.e. the definite integral.

The Riemann Sum formula provides a precise definition of the definite integral as the limit of an infinite series. The Riemann Sum formula is as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$$

Where n is the number of rectangles formed beneath the curve; $f(x_i)$ is the height of each rectangle; $x = a$ and $x = b$ are the limits of integration.

3.10. Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus justifies the procedure of evaluating the antiderivative at the upper and lower limits of integration and taking the difference. The definite integral of a function is often viewed as the area under the graph of the function between two limits and is defined as the limit of Riemann sums as the number of rectangles made to increase without bound approaching infinity (∞).

Let f be continuous at $[a,b]$. If F is an antiderivative for f on $[a,b]$, then,

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 13. Evaluate the following definite integrals using the appropriate integration formulas.

a) $\int_{-3}^{1} (6x^2 - 5x + 2)dx$

$$\int_{-3}^{1} (6x^2 - 5x + 2)dx = \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Big|_{-3}^{1}$$

$$= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) = \left(4 - \frac{5}{2}\right) - \left(-60 - \frac{45}{2}\right)$$

$$= 64 - \frac{5}{2} + \frac{45}{2} = 64 + \frac{40}{2} = 64 + 20 = 84$$

b) $\int_0^{\frac{\pi}{3}} (2\sin x - 5\cos x)dx = (-2\cos x - 5\sin x)\Big|_0^{\frac{\pi}{3}}$

$$= \left(-2\cos\frac{\pi}{3} - 5\sin\frac{\pi}{3}\right) - (-2\cos 0 - 5\sin 0)$$

$$= -2\left(\frac{1}{2}\right) - 5\left(\frac{\sqrt{3}}{2}\right) - [-2(1) - 0] = -1 - \frac{5\sqrt{3}}{2} + 2$$

$$= 1 - \frac{5\sqrt{3}}{2}$$

Example 14. Evaluate the following definite integrals using substitution rule.

a) $\int_{-1}^{0} x^3(1-2x^4)^3 dx$

Method 1: Let $u = 1 - 2x^4$, $\frac{du}{dx} = -8x^3$, $du = -8x^3dx$, $x^3dx = \frac{du}{-8}$

However, since there is change in the variable from x to u, we need to know the values of u that correspond to the given values of x. That is, when $x = 0$, $u = 1 - 2(0) = 1$ while when $x = -1$, $u = 1 - 2(-1)^4 = 1 - 2 = -1$.

Therefore, $\int_{-1}^{0} x^3(1-2x^4)^3 dx = \int_{-1}^{0} (1-2x^4)^3(x^3dx) = \int_{-1}^{1} u^3\left(\frac{du}{-8}\right) = -\frac{1}{8}\int_{-1}^{1} u^3du = -\frac{1}{8}\left(\frac{u^4}{4}\right)\Big|_{-1}^{1}$

$$= -\frac{1}{32}(1)^4 - \left(-\frac{1}{32}\right)(-1)^4 = -\frac{1}{32} + \frac{1}{32} = 0$$

Method 2: Under this method, there is no need to find the corresponding values of u. See how it is done below.

$$\int_{-1}^{0} x^3(1-2x^4)^3 dx = \int_{-1}^{0} (1-2x^4)^3(x^3dx) = -\frac{1}{8}\int_{x=-1}^{x=1} u^3(du) = -\frac{1}{8}\left(\frac{u^4}{4}\right)\Big|_{x=-1}^{x=1}$$

$$= -\frac{1}{32}(1-2x^4)^4\Big|_{x=-1}^{x=1} = -\frac{1}{32}(1-2)^4 - \left(-\frac{1}{32}\right)[1-2(-1)^4]^4$$

$$= -\frac{1}{32}(1) + \frac{1}{32}[1-2(1)]^4 = -\frac{1}{32} + \frac{1}{32}(-1)^4 = -\frac{1}{32} + \frac{1}{32} = 0$$

b) $\int_3^5 \frac{4x}{2-8x^2}dx$

Method 1. Let $u = 2 - 8x^2$, $\frac{du}{dx} = -16x$, $xdx = \frac{du}{-16}$.

Again, since there is a change in variable, we find the values of u corresponding to the given values of x. Hence, when $x = 3$, $u = 2 - 8(3)^2 = 2 - 72 = -70$ while when $x = 5$, $u = 2 - 8(5)^2 = 2 - 8(25) = 2 - 200 = -198$.

Substitute. $$\int_3^5 \frac{4x}{2-8x^2}dx = 4\int_3^5 \frac{1}{2-8x^2}(xdx) = 4\int_{-70}^{-198}\frac{1}{u}\left(-\frac{du}{16}\right) = -\frac{1}{4}\ln|u|\Big|_{-70}^{-198}$$

$$= \frac{1}{4}\ln|-198| - \frac{1}{4}\ln|-70| = \frac{1}{4}\ln 198 - \frac{1}{4}\ln 70 = -\frac{1}{4}[\ln 198 - \ln 70]$$

Simplify. $$= -\frac{1}{4}\ln\frac{198}{70} = -\frac{1}{4}\ln\frac{99}{35}$$

Method 2. $$\int_3^5 \frac{4x}{2-8x^2}dx = 4\int_3^5 \frac{1}{2-8x^2}(xdx) = 4\int_{x=3}^{x=5}\frac{1}{u}\left(-\frac{du}{16}\right) = -\frac{1}{4}\ln|u|\Big|_{x=5}^{x=5}$$

$$= -\frac{1}{4}\ln|2-8x^2|\Big|_{x=2}^{x=5} = -\frac{1}{4}\ln|2-8(5)^2| - \left[-\frac{1}{4}\ln(2-8(3))^2\right]$$

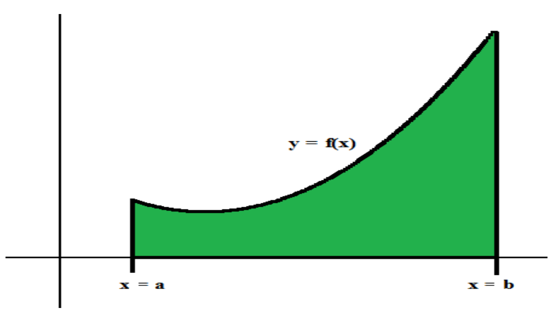
$$= -\frac{1}{4}\ln|-198| + \frac{1}{4}\ln|-70| = -\frac{1}{4}[\ln 198 - \ln 70] = -\frac{1}{4}\ln\frac{99}{35}$$

3.11. Definite Integral as Area under a Curve

The area bounded by the graph of $y = f(x)$, the x-axis and two vertical lines $x = a$ and $x = b$ is defined by the definite integral below.

$$A = \int_a^b f(x)dx$$

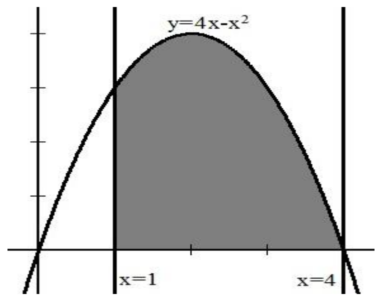
Using the above formula results to a positive area if the graph of $y = f(x)$ is above the x-axis, and a negative area if the graph is below the x-axis.
In the latter case, the actual area will be the absolute value of the calculated area.



Example 15. Find the area bounded by the given curve, the x-axis and the two vertical lines.

a) $y = 4x - x^2$, $x = 1$ and $x = 4$

Solution: The shaded region is the one bounded by the parabola $y = 4x - x^2$, the x-axis, $x = 1$ and $x = 4$. It is above the x-axis, as shown. Expectedly, the calculated area needs be positive.

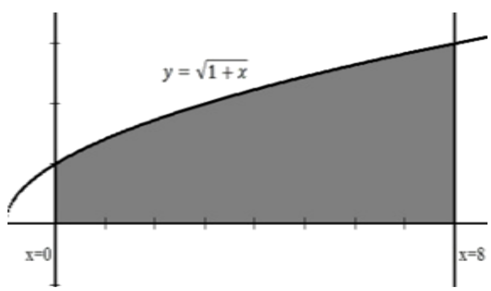


$$A = \int_1^4 f(x)dx = \int_1^4 (4x - x^2)dx = \left(2x^2 - \frac{x^3}{3}\right)\Bigg|_1^4 = \left[2(4)^2 - \frac{(4)^3}{3}\right] - \left[2(1) - \frac{1}{3}\right]$$

$$A = \left(32 - \frac{64}{3}\right) - \left(2 - \frac{1}{3}\right) = 32 - \frac{64}{3} - 2 + \frac{1}{3} = 30 - \frac{63}{3} = 30 - 21 = 9$$

b) $y = \sqrt{1 + x}$, $x = 0$ and $x = 8$

Solution: The shaded region with $y = \sqrt{1 + x}$, the x-axis, $x = 0$ and $x = 8$ as boundaries has expectedly a positive area since it is above the x-axis.



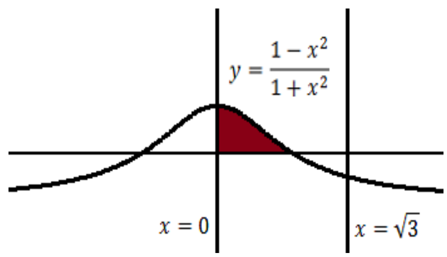
$$A = \int_0^8 f(x)dx = \int_0^8 \sqrt{1 + x}\,dx = \int_0^8 (1 + x)^{\frac{1}{2}} dx = \left[\frac{2}{3}(1 + x)^{\frac{3}{2}}\right]\Bigg|_0^8 = \frac{2}{3}\sqrt{(1 + x)^3}\Bigg|_0^8$$

$$A = \frac{2}{3}\sqrt{(1 + 8)^3} - \frac{2}{3}\sqrt{(1)^3} = \frac{2}{3}\left[\sqrt{(9)^3} - \sqrt{1}\right] = \frac{2}{3}\left[\left(\sqrt{9}\right)^3 - 1\right] = \frac{2}{3}\left[(3)^3 - 1\right]$$

$$A = \frac{2}{3}(27 - 1) = \frac{2}{3}(26) = \frac{52}{3}$$

c) $y=\dfrac{1-x^2}{1+x^2}$, $x=0$ and $x=\sqrt{3}$

Solution: The figure below shows that the region bounded by $y=\dfrac{1-x^2}{1+x^2}$, the x-axis, $x=0$ and $x=\sqrt{3}$ consists of two parts. One part is above the x-axis (hence, area is positive) from $x=0$ to $x=1$ (an x-intercept of the graph), the other one from $x=1$ to $x=\sqrt{3}$ is below, hence, area is expected to be a negative area.



Therefore, $A=A_1+A_2$.

$$A_1=\int_0^1\frac{1+x^2}{1-x^2}dx=\int_0^1\left(-1+\frac{2}{1+x^2}\right)dx=\left[-x+2\left(\frac{1}{1}\right)Tan^{-1}\left(\frac{x}{1}\right)\right]\Bigg|_0^1$$

$$A_1=\left[-1+2Tan^{-1}1\right]-\left[0+2Tan^{-1}0\right]=-1+2\left(\frac{\pi}{4}\right)=-1+\frac{\pi}{2}=0.57$$

Moreover, $$A_2=\int_1^{\sqrt{3}}\frac{1+x^2}{1-x^2}dx=\left[-x+2Tan^{-1}x\right]_1^{\sqrt{3}}=\left(-\sqrt{3}+2Tan^{-1}\sqrt{3}\right)-\left(-1+2Tan^{-1}1\right)$$

$$A_2=\left[-\sqrt{3}+2\left(\frac{\pi}{3}\right)\right]-\left[-1+2\left(\frac{\pi}{4}\right)\right]=-0.21$$

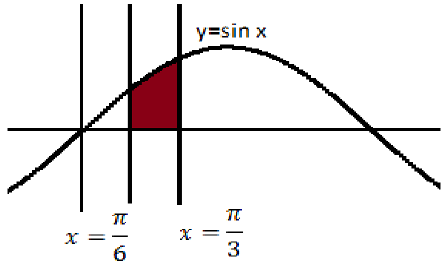
$$A_2=\left|-0.21\right|=0.21$$

Finally, the required area bounded $y=\dfrac{1-x^2}{1+x^2}$, the x-axis, $x=0$ and $x=\sqrt{3}$ is

$$A=0.57+0{,}21=0{,}78$$

d) $y=\sin x$, $x=\dfrac{\pi}{6}$ and $x=\dfrac{\pi}{3}$

Solution: The region whose area is required is below. The calculated area is positive since the region is above the x-axis.



$$A = \int_{\pi/6}^{\pi/3} \sin\, x dx = -\cos\, x \Big|_{\pi/6}^{\pi/3}$$

$$A = -\cos\frac{\pi}{3} - \left(-\cos\frac{\pi}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2}$$

Activity Sheet
INDEFINITE INTEGRAL

Name : ______________________________ **SCORE :** __________

Section :________________________**Class Schedule :** ____________________

Directions: Evaluate each of the following indefinite integral.

1. $\int (x + 25)dx$

2. $\int (x - 4)^3 dx$

3. $\int (4 + 7x)x^2 dx$

4. $\int (x - 1)(5x + 3)dx$

6. $\int (\frac{2}{x^3} + \frac{3}{8}x^8 - 10x)dx$

7. $\int (\frac{x^3 - 4x + 7}{x^5})dx$

8. $\int (x^{\frac{4}{3}} + 5x - 6)dx$

9. $\int \frac{10x^7 + 4x^6 - 3x^5 - 7x^4 + 3x^2 + 3}{x^2} dx$

5. $\int (2x-1)^2 (x-3)^3 \, dx$

10. $\int (4\sqrt{x} + \frac{5}{2\sqrt{x}}) dx$

11. $\int (3\cos x + 7\sin x) dx$

16. $\int 5^{2x+7} \, dx$

12. $\int (x\tan^3 x + 1) dx$

17. $\int (4 - 7e^{2x}) dx$

13. $\int (x^4 - 3\cos x) dx$

18. $\int (3e^x + 4e^{-x})^3 \, dx$

14. $\int \frac{\sin x}{3 - \sin^2 x} dx$

19. $\int e^{x^5} 5x^4 dx$

15. $\int 2^{3x} \, dx$

20. $\int \frac{3dx}{x \ln x}$

Activity Sheet
INTEGRATION BY SUBSTITUTION

Name : ____ **SCORE :** __________

Section :___________________________**Class Schedule :** ____________________

Directions: Evaluate the following using the appropriate substitution.

1. $\int 5(7x+1)^3\,dx$

2. $\int 3x(4x^4+5)^3\,dx$

3. $\int \frac{-6x}{(2-3x^2)^3}\,dx$

4. $\int \frac{5dx}{16+9x^2}\,dx$

6. $\int \frac{x^3+x+1}{x^3+1}\,dx$

7. $\int 2x^3\sqrt{x^4+2}\,dx$

8. $\int (x+4)\sqrt{x^2+6x+5}\,dx$

9. $\int \sqrt{4x-7}\,dx$

5. $\int \frac{4x^3+2}{x^4+2x}dx$

10. $\int 2x\sqrt{4x-5}dx$

11. $\int \cos^2 x \sin x dx$

16. $\int e^{4x-5}dx$

12. $\int \cot 5x \csc 5x dx$

17. $\int \left(5e^{6x}+3\right)dx$

13. $\int \sin^3 4x \cos 4x dx$

18. $\int e^x \csc^3(e^x)dx$

14. $\int 4\sec^2 x(\tan x+2)dx$

19. $\int \frac{\sin(\ln x)}{x}dx$

15. $\int \frac{\sin 3x}{2+\cos^3 x}dx$

20. $\int e^{\tan x}\sec^2 x dx$

Activity Sheet
DIFFERENTIAL EQUATION

Name : ______________________________ **SCORE :** __________

Section :________________________**Class Schedule :** __________________

A. Verify that the general solution of the differential equation satisfies the differential equation.

1. $y = Ce^{3x}$; $y'' + 2y' - 15y = 0$

2. $5x^2 + 4y^2 = C$; $4yy' + 5x = 0$

3. $y = C_1e^x + C_2xe^x$; $y'' - 2y' + y = 0$

4. $y = Cx^2e^x$; $xy' - 2y = x^3e^x$

5. $3e^x + e^y - 6x = C$; $e^y y' = 6 - 3e^x$

B. Find the particular solution of each differential equation given the conditions.

1. $y' = 12x^2 + 8x + 1$; y = 150 when x = 1

2. $y' = 16x^3 + 8x$; y = 5 when x = 0

3. $y' = \frac{4}{x^3} - \frac{2}{x^2} + \frac{1}{x} + 2$; y = 2 when x = 5

4. $y' = \frac{x^3}{x^3 - 1}$; y = 3 when x = 0

5. $y' = x^2\sqrt{x^2 - 25}$; y=1 when x = 5

Activity Sheet
DIFFERENTIAL EQUATION

Name : ________________________________ **SCORE :** ____________

Section :____________________**Class Schedule :** ________________

Directions: Solve each of the following problems

1. The number of fungi in a culture is increasing exponentially. There are 400 fungi in the culture after 30 minutes and had 700 fungi after 3 hours. How many fungi are there initially? After how many hours will there be 900 fungi in the culture?

2. The population of certain dragonflies is growing according to the logistic equation. Consider that the maximum population of these dragonflies is 300. The number of dragonflies increased from 70 to 90 for a period of 3 months. In how many months will the number of dragonflies reach its maximum population size?

Activity Sheet
DEFINITE INTEGRALS

Name : ______________________________ **SCORE :** __________

Section :______________________ **Class Schedule :** ________________

Directions: Evaluate each of the following definite integrals

1. $\int_{-3}^{7}(5x-7)dx$

2. $\int_{-4}^{4}\left(x^3+5x-3\right)dx$

3. $\int_{-1}^{2}\left(\frac{1}{4}x^4-6x^3-2x+7\right)dx$

4. $\int_{0}^{4}\left(5x^2+7\right)^4 dx$

6. $\int_{-1}^{3}2x\left(3x^3+4x^2-7\right)dx$

7. $\int_{-1}^{2}(2x+5)(x-3)^2 dx$

8. $\int_{1}^{4}\left(x^3-9\right)dx$

9. $\int_{-3}^{-2}\left(\frac{7}{x^3}-\frac{5}{3}x^4+2x-5\right)dx$

5. $\int_{2}^{5} (x+3)(x^2-2)dx$

11. $\int_{0}^{\frac{\pi}{4}} \sec^2 x dx$

12. $\int_{0}^{\pi} (1+\cos x)dx$

13. $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{1-\sin^2 x}{\cos^2 x} dx$

14. $\int_{0}^{\pi} 4 \tan x \sec x dx$

15. $\int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 x}{3+\tan x} dx$

10. $\int_{4}^{9} 5\sqrt{x} dx$

16. $\int_{\frac{\pi}{4}}^{\pi} (x+\cos x)dx$

17. $\int_{1}^{2} \sec^2 x e^{\tan x} dx$

18. $\int_{-1}^{1} e^{-x} dx$

19. $\int_{e}^{e^2} \frac{6}{x \ln x} dx$

20. $\int_{1}^{e} \frac{(2+\ln x)^2}{x} dx$

Activity Sheet
AREA OF A REGION

Name : ______________________________ **SCORE :** ____________

Section :______________________**Class Schedule :** ________________

Directions: Find the area of the region bounded by the graphs of the equations.

1. $y = x^2 - 16, y = 0, x = 1, x = 3$

2. $y = -x^2 + 6x, y = 0$

3. $y = (5 - x)\sqrt{x}, y = 0$

4. $y = \dfrac{3}{\sqrt{9 - x^2}}, y = 0, x = 1, x = 2$

5. $y = \sin x, x = 0, x = \dfrac{2\pi}{3}$

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